



This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

### Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

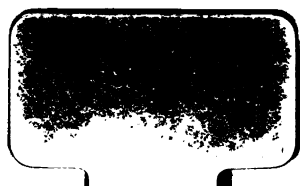
We also ask that you:

- + *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.
- + *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.
- + *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.
- + *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

### About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at <http://books.google.com/>







1

2

3



# ELEMENTARY GEOMETRY.

BOOKS I.—V.

“Ὡς οὖν τ’ ἄρα, ἦν δ’ ἐγώ, μάλιστα προστακτέον, ὅπως οἱ ἐν τῇ καλλι-  
πόλει σοι μηδενὶ τρόπῳ γεωμετρίας ἀφείζονται· πρὸς γὰρ πάσας μαθήσεις,  
ὥστε καλλίον ἀποδέχεσθαι, ἴσμεν πού οἱ τῷ ὅλῳ καὶ παντὶ διοίσει ἡμμένο-  
τε γεωμετρίας καὶ μή. τῷ παντὶ μέντοι νῆ Δί’, ἔφη.”

PLATO, *Repub.* Bk. VII. 527.

This was Divine Plato his Judgement, both of the purposed, chief, and perfect use of Geometrie; and of his second, depending and derivative commodities. And for us, Christen men, a thousand thousand occasions are to have neede of the helpe of Metaphologicall Contemplations; whereby to trayne our Imaginations and Mynes, by litle and litle, to forsake and abandon the grosse and corruptible Objectes of our outward senses: and to apprehend, by sure Doctrine demonstrative, Things Mathematicall.

---

John Dee his Mathematicall Preface to Euclides Elements.

A. D. 1570.

# ELEMENTARY GEOMETRY.

BOOKS I.—V.

CONTAINING

THE SUBJECTS OF EUCLID'S FIRST SIX BOOKS:

FOLLOWING THE SYLLABUS OF GEOMETRY PREPARED  
BY THE GEOMETRICAL ASSOCIATION.

BY

J. M. WILSON, M.A.

LATE FELLOW OF ST JOHN'S COLLEGE, CAMBRIDGE,  
AND MATHEMATICAL MASTER OF RUGBY SCHOOL.

FOURTH EDITION.



London:

MACMILLAN AND CO.

1878.

[The right of translation and reproduction is reserved.]

183 . 2 . 102 .



**Cambridge :**  
**PRINTED BY C. J. CLAY, M.A.**  
**AT THE UNIVERSITY PRESS.**

## INTRODUCTION.

THE Science of *Geometry* treats of the properties and construction of *solids*, *surfaces*, and *lines*. *Plane* Geometry treats only of the *line* and *plane* or flat surface; and the *elements* of Plane Geometry include the properties of the *straight* line and *circle* only, and of combinations of straight lines and circles.

The science of Geometry is called *deductive*, because certain fundamental truths being assumed as obviously true, the remaining truths of the science are deduced from them by reasoning.

Propositions admitted without demonstration are called *Axioms*.

Of the Axioms used in Geometry those are termed *General* which are applicable to magnitudes of all kinds: the following is a list of the general axioms more frequently used.

- (a) The whole is greater than its part.
- (b) The whole is equal to the sum of its parts.
- (c) Things that are equal to the same thing are equal to one another.
- (d) If equals are added to equals the sums are equal.
- (e) If equals are taken from equals the remainders are equal.

(f) If equals are added to unequals the sums are unequal, the greater sum being that which is obtained from the greater magnitude.

(g) If equals are taken from unequals the remainders are unequal, the greater remainder being that which is obtained from the greater magnitude.

(h) The doubles and halves of equals are equal.

A *Theorem* is the formal statement of a proposition that may be demonstrated from known propositions. These known propositions may themselves be Theorems or Axioms.

The two next pages, within brackets, may be omitted the first time of reading the subject.

[A Theorem consists of two parts, the *hypothesis*, or that which is assumed, and the *conclusion*, or that which is asserted to follow therefrom. Thus in the typical Theorem

*If A is B, then C is D, (i)*

the hypothesis is that A is B, and the conclusion, that C is D.

From the truth conveyed in this Theorem it necessarily follows :

*If C is not D, then A is not B, (ii).*

Two such Theorems as (i) and (ii) are said to be *contrapositive*, each of the other.

For example, if it were universally true that, If a man is a Spaniard, his hair is black ; then it would follow that if his hair is not black, the man is not a Spaniard. Each of these statements is the contrapositive of the other.

Two Theorems are said to be *converse*, each of the other, when the hypothesis of each is the conclusion of the other.

Thus,

*If C is D, then A is B, (iii)*

is the converse of the typical Theorem (i).

The contrapositive of the last Theorem, viz. :

*If A is not B, then C is not D, (iv)*

is termed the *obverse* of the typical Theorem (i).

Sometimes the hypothesis of a Theorem is complex, i.e. consists of several distinct hypotheses ; in this case every Theorem formed by interchanging the conclusion and *one* of the hypotheses is a converse of the original Theorem.

The truth of a converse is not a logical consequence of the truth of the original Theorem, but requires independent investigation.

Thus, supposing it were true that if a man is a Spaniard his hair is black ; it does not follow that if a man's hair is black he is therefore a Spaniard : for he might be a Turk or of any other nation.

Hence the four associated Theorems (i) (ii) (iii) (iv) resolve themselves into two Theorems that are independent of one another, and two others that are always and necessarily true if the former are true ; consequently it will never be necessary to demonstrate *geometrically* more than two of the four Theorems, care being taken that the two selected are not contrapositive each of the other.

*Rule of Conversion.* If of the hypotheses of a group of demonstrated Theorems it can be said that one must be true, and of the conclusions that no two can be true at the same time, then the converse of every Theorem of the group will necessarily be true.

Obs. The simplest example of such a group is presented when a Theorem and its obverse have been demonstrated, and the validity of the rule in this instance is obvious from the circumstance that the converse of each of two such Theorems is the contrapositive of the other. Another example, of frequent occurrence in the elements of Geometry, is of the following type :

*If A is greater than B, C is greater than D.*

*If A is equal to B, C is equal to D.*

*If A is less than B, C is less than D.*

Three such Theorems having been demonstrated *geometrically*, the converse of each is always and necessarily true.

*Rule of Identity.* If there is but one A, and but one B ; then from the fact that A is B it necessarily follows that B is A.

This is an important axiom in geometrical reasoning. De Morgan used to illustrate it by the following example :—

Suppose that in a town there were *only one* post-office and *only one* grocer's ; and that it was known that the post-office was the grocer's ; then it would follow that the grocer's was the post-office.

This is called the axiom of the unique solution, or the rule of identity.]

#### EXPLANATION OF TERMS AND SIGNS.

A *Problem* is a geometrical construction to be effected by the aid of certain instruments.

It has been universally agreed by Geometers to use *only the ruler*, i. e. a straight edge, not divided, and a *pair of compasses*, in the solution of Problems.

A *Corollary* is a geometrical truth easily deducible from a theorem.

Q. E. D. stands for *quod erat demonstrandum*, and is usually written at the end of a theorem to mark that the truth of the theorem has been proved.

The parts of a Theorem are the *general enunciation* of the hypothesis and the fact to be proved, or statement in general language; the *particular enunciation*, or statement of the hypothesis and the fact to be proved in the particular case examined; and the *proof*.

In the proof it is frequently necessary to draw certain lines, or to conceive them as drawn. This is called the *construction*.

## REMARKS.

A beginner often asks 'What is the use of Geometry?'

The following remarks may perhaps help to shew him part at least of the use of it.

What is Geometry? What is the object of the science?

It is not *measurement*, because that may be done *directly*. If I want to find the height of a tower, I may go to the top, and let a string down to the bottom, and then measure the string; but this is not geometry, though it is measurement. Geometry is the science of *indirect measurement*; in which, for example, by measuring one line we learn the length of another. If I measure the length of the *shadow* of the tower, and also the length of a vertical stick and its shadow, and have proved by geometrical reasoning, that as the length of shadow of the stick is to the length of shadow of the tower, so is the height of the stick to the height of the tower, that is, *measure the height of the tower indirectly*, this is a geometrical operation.

Now it is plain that many measurements *must* be effected indirectly. How for example is the height of a mountain ascertained? Or how is the distance of the moon from the earth found out to be very nearly 238000 miles? How do we know approximately the size of the sun, or the weight of some of the stars, or the velocity of light? It is plain that these results must be obtained by indirect measurement; and some of

them are obtained by measurement extremely indirect and circuitous, and consisting of a very great number of successive steps of reasoning; each result, as soon as it is obtained, serving as the starting-point from which fresh results are attainable.

Now Elementary Geometry gives the beginning of all such chains of reasoning. The theorems are results which follow from the axioms, and, in their turn, will serve as the foundations for fresh theorems arranged in a long chain until questions such as those above mentioned can be solved.

Every theorem therefore may be shewn to be a means of indirectly measuring some magnitude. In theorem 4, for example, it is proved that  $AOD = COB$ ; that is, if  $AOD$  is accessible, and is measured (by an instrument suitable for measuring angles), then it is not necessary to measure  $COB$ , for you have proved that it will be the same as  $AOD$ .

Again in Theorem 7, let  $A$  be a post on one bank of a river,  $B$ ,  $C$  two posts on the opposite bank; it is required to find the distance across the river from  $B$  to  $A$ .

Measure  $BC$ , (which you can do, as they are both on the same bank,) and put up two posts  $E$ ,  $F$  in a field at the same distance apart that  $B$  is from  $C$ : measure the angle at  $B$ , that is how much, when standing at  $B$ , you must turn a line pointing at  $C$  till it points at  $A$ ; and copy this angle at  $E$ : similarly measure the angle at  $C$ , and copy it at  $F$ . Then this theorem has proved that  $AB = DE$ ; that is if you measure in the field  $ED$ , you will indirectly have measured  $AB$ .

Theorem 5 is of very great importance, and is a good illustration of indirect measurement. Suppose  $B$  and  $C$  are two points with an obstacle between them, a house or a hill for example; how is the distance from  $B$  to  $C$  to be measured? This theorem tells you; you may think it out for yourself.

So with this clue to the practical application of the theorems it will be well to go through all of them; finding out in each case what the magnitude is which is indirectly measured, or the result indirectly obtained; and inventing practical questions to which each theorem could be applied.

## BOOK I.

### THE STRAIGHT LINE.

#### DEFINITIONS.

*Def. 1.* A *point* has position, but it has no magnitude.

*Def. 2.* A *line* has position, and it has length, but has neither breadth nor thickness. The extremities of a line are points, and the intersection of two lines is a point.

*Def. 3.* A *surface* has position, and it has length and breadth, but not thickness. The boundaries of a surface, and the intersection of two surfaces, are lines.

*Def. 4.* A *solid* has position, and it has length, breadth and thickness.

The boundaries of a solid are surfaces.

*Def. 5.* A *straight line* is such that any part will, however placed, lie wholly on any other part, if its extremities are made to fall on that other part.

*Def. 6.* A *plane surface*, or *plane*, is a surface in which any two points being taken the straight line that joins them lies wholly in that surface.



*Def. 7.* A *plane figure* is a portion of a plane surface enclosed by a line or lines.

*Def. 8.* A *circle* is a plane figure contained by one line, which is called the *circumference*, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another. This point is called the *centre* of the circle.

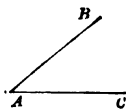
*Def. 9.* A *radius* of a circle is a straight line drawn from the centre to the circumference.

*Def. 10.* A *diameter* of a circle is a straight line drawn through the centre and terminated both ways by the circumference.

*Def. 11.* When two straight lines are drawn from the same point, they are said to contain, or to make with each other, a *plane angle*. The point is called the *vertex*, and the straight lines are called the *arms*, of the angle.

An *angle* is a simple concept incapable of *definition*, properly so called, but the nature of the concept may be explained as follows, and for convenience of reference it may be reckoned among the definitions.

A line drawn from the vertex and turning about the vertex in the plane of the angle from the position of coincidence with one arm to that of coincidence with the other is said to turn through the angle: and the angle is greater as the quantity of turning is greater. Since the line may turn from the one position to the other in either of two ways, two angles are formed by two straight lines drawn from a point. These angles (which have a common vertex and common arms) are said to be *conjugate*. The greater of the two is called the *major conjugate*, and the smaller the *minor conjugate*, angle.



When *the angle contained by two lines* is spoken of without qualification, the *minor conjugate* angle is to be understood. It is seldom requisite to consider major conjugate angles before Book III.

When the arms of an angle are in the same straight line, the conjugate angles are equal, and each is then said to be a *straight angle*.

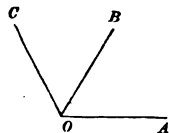
An angle is named by a single letter at its vertex, as  $A$  : or by a letter at the vertex placed between letters at points on each of its arms, as  $BAC$ , or  $CAB$ .

*Def. 12.* When three straight lines are drawn from a point, if one of them be regarded as lying between the other two, the angles which this one (the mean) makes with the other two (the extremes) are said to be *adjacent angles* : and the angle between the extremes, through which a line would turn in passing from one extreme through the mean to the other extreme, is the sum of the two adjacent angles.

Thus  $AOB$ ,  $BOC$  are adjacent,

and  $AOB + BOC = AOC$ ,

also  $AOC - COB = AOB$ .



*Def. 13.* The *bisector* of an angle is the straight line that divides it into two equal angles.

*Def. 14.* When one straight line stands upon another straight line and makes the adjacent angles equal, each of the angles is called a *right angle*.

OBS. Hence a straight angle is equal to two right angles ; or, a right angle is half a straight angle, and a straight line makes with its continuation at any point an angle of two right angles.

*Def. 15.* A *perpendicular* to a straight line is a straight line that makes a right angle with it.

*Def. 16.* An *acute* angle is that which is less than a right angle.

*Def. 17.* An *obtuse* angle is that which is greater than one right angle, but less than two right angles.

*Def. 18.* A *reflex* angle is a term sometimes used for a major conjugate angle.

*Def. 19.* When the sum of two angles is a right angle, each is called the *complement* of the other, or is said to be *complementary* to the other.

*Def. 20.* When the sum of two angles is two right angles, each is called the *supplement* of the other, or is said to be *supplementary* to the other.

*Def. 21.* The opposite angles made by two straight lines that intersect are called *vertically opposite angles*.

*Def. 22.* A *plane rectilineal figure* is a portion of a plane surface inclosed by straight lines. When there are more than three inclosing straight lines the figure is called a *polygon*.

*Def. 23.* A polygon is said to be *convex* when no one of its angles is reflex.

*Def. 24.* A polygon is said to be *regular* when it is equilateral and equiangular; that is, when all its sides and angles are equal.

*Def. 25.* A *diagonal* is the straight line joining the vertices of any angles of a polygon which have not a common arm.

*Def. 26.* The *perimeter* of a rectilineal figure is the sum of its sides.

*Def. 27.* The *area* of a figure is the space inclosed by its boundary.

*Def. 28.* A *triangle* is a figure contained by three straight lines.

*Def. 29.* A *quadrilateral* is a polygon of four sides, a *pentagon* one of five sides, a *hexagon* one of six sides, and so on.

### GEOMETRICAL AXIOMS.

1. Magnitudes that can be made to coincide are equal.
2. Two straight lines that have two points in common lie wholly in the same straight line.
3. A finite straight line has one and only one point of bisection.
4. An angle has one and only one bisector.

### POSTULATES.

Let it be granted that

1. A straight line may be drawn from any one point to any other point.
2. A terminated straight line may be produced to any length in a straight line.
3. A circle may be described from any centre, with a radius equal to any finite straight line.

It will be seen that these postulates amount to a request to use the straight edge of a ruler, and a pair of compasses ; the latter being such that a distance can be carried by them from one part of the paper to another.

It may be useful to have a list of the derivations of some of the common terms used in geometry :

*Axiom.* ἀξίωμα, a statement deemed true.

*Theorem.* θεώρημα.

*Hypothesis.* ὑπόθεσις, a supposition, a foundation, from ὑπὸ, τίθημι.

*Identity.* Idem, identidem, the same thing.

*Geometry.* γῆ, μετρέω, to measure land.

*Diameter.* διὰ μετρέω, to measure across.

*Plane.* Planus, level.

*Complement.* Compleo, to fill up.

*Rectilinear.* Recta, linea, a straight line.

*Polygon.* πολλῖς, γωνία, many angles.

*Diagonal.* διὰ γωνία, across from angle to angle.

*Perimeter.* περὶ μετρέω, to measure round.

*Triangle.* Tres, angulus, with three angles.

*Quadrilateral.* Quadra (quater), latus, with four sides.

*Equilateral.* Æquus, latus, having equal sides.

*Pentagon.* πέντε γωνία, with five angles.

*Postulate.* Postulatum, a thing requested.

*Isosceles.* ἴσος σκέλος, ἰσοσκελής, having equal legs.

*Hypotenuse.* ὑπὸ τείνουσα (γραμμὴ), the line stretching across.

*Parallel.* παρὰ, ἀλλήλα, alongside of one another.

*Parallelogram.* παρὰ, ἀλλήλα, γραμμὴ, made by lines alongside of one another.

*Trapezium.* τράπεζα, a table.

*Trapezoid.* τραπεζοειδής, like a table.

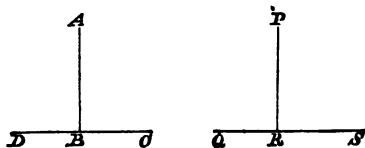
*Orthogonal.* ὀρθός, γωνία, having right angles.

## SECTION I.

### ANGLES AT A POINT.

#### THEOREM I.

*All right angles are equal to one another.*



*Part. En.* Let  $ABC$ ,  $PRS$  be right angles ;  
it is required to prove that  $ABC$  is equal to  $PRS$ .

*Proof.* If the point  $B$  were placed on the point  $R$ , and the line  $BC$  along the line  $RS$ ,

then because the lines  $DC$ ,  $QS$  are *straight*,  
therefore the line  $BD$  would fall along the line  $RQ$ ;

(Ax. 2.)

therefore the angle  $DBC$  coincides with, and is equal to, the  
angle  $QRS$ .

(Ax. 1.)

But by Def. 14, the angle  $ABC$  is half the angle  $DBC$ ;  
and the angle  $PRS$  is half the angle  $QRS$ ;  
and the halves of equals are equal;  
therefore the right angle  $ABC$  is equal to the right angle  
 $PRS$ .

Q. E. D.

COR. 1. *At a given point in a given straight line there  
can be only one perpendicular drawn to that line.*

COR. 2. *The complements of equal angles are equal.*

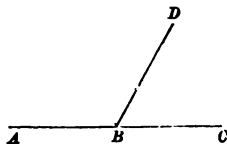
COR. 3. *The supplements of equal angles are equal.*

## THEOREM 2.

*If a straight line stands upon another straight line it  
makes the adjacent angles together equal to two right angles\*.*

*Part. En.* Let  $DB$  stand upon  
the straight line  $AC$ ;

it is required to prove that the ad-  
jacent angles  $ABD$ ,  $DBC$  are to-  
gether equal to two right angles.



*Proof.* Because  $ABC$  is a *straight* line, (Hyp.)  
therefore the angle  $ABC$  is equal to two right angles;

(Def. 14.)

but the angle  $ABC$  is, from the figure, made up of the angles  
 $ABD$  and  $DBC$ ;

(Def. 12.)

therefore the angles  $ABD$  and  $DBC$  are together equal to  
two right angles.

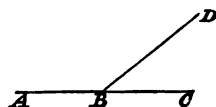
Q. E. D.

\* Eucl. I. 13.

## THEOREM 3.

*If the adjacent angles made by one straight line with two others are together equal to two right angles, these two straight lines are in one straight line\*.*

*Part. En.* Let the adjacent angles  $DBA$ ,  $DBC$  made by  $BD$  with the two straight lines  $BA$ ,  $BC$  be together equal to two right angles;



it is required to prove that  $AB$ ,  $BC$  are in one straight line.

*Proof.* Because  $DBA$  and  $DBC$  are together equal to two right angles; (Hyp.)

and  $DBA$  and  $DBC$ , from the figure, make up the angle  $ABC$ ;

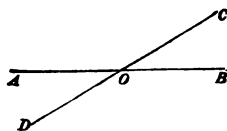
therefore  $ABC$  is an angle of two right angles;

and therefore  $ABC$  is a straight line. (Def. 14.) Q. E. D.

## THEOREM 4.

*If two straight lines cut one another the vertically opposite angles will be equal to one another.*

*Part. En.* Let the straight lines  $AOB$ ,  $DOC$  cut one another, and let  $AOD$ ,  $BOC$  be vertically opposite angles;



it is required to prove that the angle  $AOD$  is equal to the angle  $BOC$ .

*Proof.* Because  $AOB$  is a straight line; (Hyp.)  
therefore the angles  $AOC$  and  $COB$  are together equal to two right angles. (Th. 2.)

\* Eucl. I. 14.

And again because  $DOC$  is a *straight* line ; (Hyp.)  
therefore the angles  $AOC$  and  $AOD$  are together equal to  
two right angles. (Th. 2.)

Therefore the angles  $AOC$  and  $COB$  are equal to the  
angles  $AOC$  and  $AOD$ .

Take away the common angle  $AOC$  ;  
therefore the angle  $COB$  is equal to the angle  $AOD$ .\*

Q. E. D.

COR. *The sum of all the angles made by any number of  
lines taken consecutively which meet at a point is four right  
angles†.*

#### EXERCISES ON ANGLES.

1. If two straight lines intersect at a point, and one of the four angles is a right angle, prove that the other three are right angles.
2. *Two* straight lines meet at a point. Are the angles at that point together equal to four right angles ?
3. If the four angles made by four straight lines which meet at a point are all right angles, prove that the four lines form two straight lines.
4. If five lines meet at a point and make equal angles with one another all round that point, each of the angles is four-fifths of a right angle.
5. Of two supplementary angles the greater is double of the less ; find what fraction the less is of four right angles.
6. Twelve lines meet at a point so as to form a regular twelve-rayed star : find the angle between consecutive rays.

\* Eucl. I. 15.

† Eucl. I. 15. Cor.



7. If four straight lines  $OA$ ,  $OB$ ,  $OC$ ,  $OD$  meet at a point, and  $AOB = COD$ , and  $BOC = DOA$ , prove that  $AOC$ ,  $BOD$  are straight lines.

8. Prove that the bisectors of adjacent supplementary angles are at right angles to one another.

9. Find the angle between the bisectors of adjacent complementary angles.

10. Prove that the bisectors of the four angles which one straight line makes with another form two straight lines at right angles to one another.

11. If four lines  $AO$ ,  $BO$ ,  $CO$ ,  $DO$  meet at a point  $O$ , and the angles  $AOB$ ,  $COD$  are given equal, and also  $AO$ ,  $CO$  are given as being in the same straight line; prove that  $BO$  and  $DO$ , if on opposite sides of  $AOC$ , are also in the same straight line.

12. If the corner of the page of a book be folded down so as to form an oblique crease, prove that the bisector of the angle between the parts of the edge that meet at the crease will be at right angles to the crease.

## QUESTIONS ON SECTION I.

7

1. What is meant by the Elements of Plane Geometry?
2. Explain the terms *axiom*, *theorem*, *converse*, *contrapositive*, giving examples of each.
3. State the Geometrical Axioms.
4. What is meant by the axiom of the rule of Identity?
5. State the fact that "all geese have two legs" in the form of a theorem, with hypothesis and conclusion; and write down its obverse, converse, and contrapositive theorems.
6. Define a plane surface, and give the test by which a surface is ascertained to be or not to be plane.
7. On what does the magnitude of an angle depend? Shew that its magnitude does not depend on the length of the arms.
8. What is meant by saying that two points *determine* a straight line?
9. What are *adjacent* angles; *supplementary* angles; *reflex* angles?
10. Shew how to find the sum and difference of two straight lines: and prove that their sum and difference together are double of the greater of the two straight lines.
11. Given the sum and difference of two straight lines; find the lengths of the straight lines.
12. Enunciate and prove the obverse and converse of Theorem 4.

## SECTION II.

## TRIANGLES.

*Def. 30.* An *isosceles* triangle is that which has two sides equal.

*Def. 31.* A *right-angled* triangle is that which has one of its angles a right angle. An *obtuse-angled* triangle is that which has one of its angles an obtuse angle. All other triangles are called *acute-angled triangles*.

*Def. 32.* A triangle is sometimes regarded as standing on a selected side which is then called its *base*, and the intersection of the other two sides is called the *vertex*. When two of the sides of a triangle have been mentioned, the remaining side is often called the *base*.

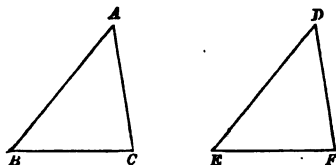
*Def. 33.* The side of a right-angled triangle which is opposite to the right angle is called the *hypotenuse*.

*Def. 34.* Figures that may be made by superposition to coincide with one another are said to be *identically* equal; and every part of one being equal to a corresponding part of the other, they are said to be equal in all respects.

## THEOREM 5.

*If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by these sides equal, then the triangles are identically equal, and of the angles those are equal which are opposite to the equal sides.*

*Part. En.* Let the two triangles  $BAC$ ,  $EDF$  have two sides of the one equal to two sides of the other, each to each, and likewise the included angles equal, viz.



$$BA = ED,$$

$$AC = DF,$$

and the included angle  $BAC =$  the included angle  $EDF$ ; }  
it is required to prove that the triangles are equal in all respects,

viz. the base  $BC$  equal to the base  $EF$ , and the angle  $B$  to the angle  $E$ , and the angle  $C$  to the angle  $F$ , and the area  $ABC$  to the area  $DEF$ .

*Proof.* If the point  $A$  be placed on the point  $D$ ,  
and the line  $AB$  were placed along  $DE$ ,  
then because the angle  $BAC =$  the angle  $EDF$ , (Hyp.)  
therefore the line  $AC$  would lie along  $DF$ .

And because  $AB = DE$ , (Hyp.)  
therefore the point  $B$  would coincide with the point  $E$ .

And because  $AC = DF$ ; (Hyp.)  
therefore the point  $C$  would coincide with the point  $F$ .

Therefore  $BC$  would coincide with  $EF$ , (Ax. 2.)

and therefore  $BC = EF$ ; (Ax. 1.)

and the angles  $B$  and  $C$  respectively coincide with and are equal to the angles  $E$  and  $F$ ,

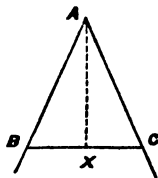
and the area of the triangle  $BAC$  coincides with and is equal to the area of the triangle  $EDF^*$ . Q. E. D.

\*. Eucl. I. 4.

## THEOREM 6.

*The angles at the base of an isosceles triangle are equal to one another\*.*

*Part. En.* Let  $ABC$  be an isosceles triangle, having the side  $AB$  equal to the side  $AC$ ;  
it is required to prove that the angle  $B$  is equal to the angle  $C$ .



*Proof.* Let  $AX$  be the bisector of the angle  $BAC$ ,  
(Ax. 4.)  
meeting the base  $BC$  in  $X$ .

Then in the triangles  $BAX$ ,  $CAX$  we have

$$BA = AC, \quad (\text{Hyp.})$$

$$AX \text{ common,}$$

and the included angle  $BAX =$  the included angle  $CAX$ .

(Hyp.)

Therefore the triangles are equal in all respects, (Th. 5.)  
that is, the angle at  $B =$  the angle at  $C$ . Q. E. D.

**COR. 1.** *If the equal sides be produced the angles on the other side of the base will be equal.*

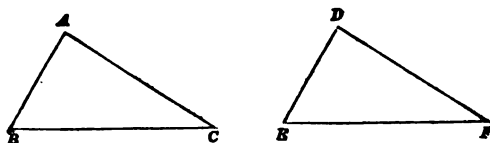
**COR. 2.** *If a triangle is equilateral it is also equiangular.*

\* Eucl. I. 5.

## THEOREM 7.

*If two triangles have one side of the one equal to one side of the other, and the angles at the extremities of those sides equal, each to each, then the triangles are equal in all respects, those sides being equal which are opposite to the equal angles\*.*

*Part. En.* Let the triangles  $ABC$ ,  $DEF$  have



$BC = EF,$

the angle  $B =$  the angle  $E,$   
and the angle  $C =$  the angle  $F;$  }

it is required to prove that the triangles are equal in all respects.

*Proof.* For if the point  $B$  were placed on the point  $E$ , and the line  $BC$  along the line  $EF$ ;

then because  $BC = EF,$  (Hyp.)  
therefore the point  $C$  would fall on  $F$ .

And because the angle  $B =$  the angle  $E,$  (Hyp.)  
therefore the line  $BA$  would fall along the line  $ED$ .

And because the angle  $C =$  the angle  $F,$  (Hyp.)  
therefore the line  $CA$  would fall along the line  $FD$ ;  
therefore the point  $A$  would fall on the point  $D$ ,  
since two straight lines can intersect in one point only;

(Ax. 2.)  
and therefore the triangles coincide and are equal in all respects,  $AB$  being equal to  $DE$ ,  $AC$  to  $DF$ , and the

\* Eucl. I. 26. Part 1.

angle  $A$  to the angle  $D$ , and the area  $ABC$  to the area  $DEF$ .

Q. E. D.

### THEOREM 8.

*If the angles at the base of a triangle are equal to one another, the triangle is isosceles\*.*

*Part. En.* Let the two angles  $B$  and  $C$  of the triangle  $ABC$  be equal; it is required to prove that  $AB = AC$ .



*Proof.* If the triangle were taken up and reversed and replaced, so that the point  $C$  fell where  $B$  was, and the line  $CB$  along the line  $BC$ , then  $B$  would fall where  $C$  was.

And because the angle  $C =$  the angle  $B$ , (Hyp.) the line  $CA$  would lie along  $BA$ , and  $BA$  along  $CA$ ; therefore the point  $A$  would coincide with its former position, and the lines  $AC$ ,  $AB$  would coincide with the lines  $AB$ ,  $AC$ .

Therefore  $AB = AC$ .

Q. E. D.

*COR.* If a triangle is equiangular, it is also equilateral.

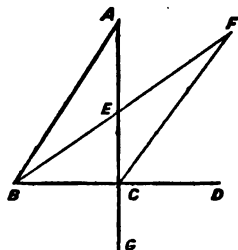
### THEOREM 9.

*If any side of a triangle be produced, the exterior angle will be greater than either of the interior and opposite angles.*

*Part. En.* Let  $ABC$  be a triangle, and let one of its sides  $BC$  be produced to  $D$ ; it is required to prove that the exterior angle  $ACD$  is greater than either of the interior and opposite angles  $CAB$  or  $ABC$ .

\* Euclid, I. 6.

*Proof.* Firstly to prove that  $ACD$  is greater than  $BAC$ .  
Let  $AC$  be bisected in  $E$ . (Ax. 3.)



Join  $BE$ , and produce it to  $F$ , making  $EF = EB$ . And join  $FC$ .

Then in the triangles  $AEB$ ,  $CEF$  we have

$$\begin{array}{ll} AE = EC, & (\text{Constr.}) \\ BE = EF, & (\text{Constr.}) \end{array}$$

and the contained angles  $AEB$ ,  $CEF$  are equal; (Th. 4.)  
therefore the triangles are equal in all respects; (Th. 5.)  
and therefore the angle  $EAB =$  the angle  $ECF$ .

But the angle  $ECD$  is greater than  $ECF$ ;  
therefore the angle  $ECD$  is also greater than  $EAB$ .

Again, if  $AC$  is produced to  $G$ , and  $BC$  is bisected, it may be similarly shewn that  $BCG$  is greater than  $ABC$ :

but  $BCG$  is equal to  $ACD$ ,

therefore  $ACD$  is also greater than  $ABC$ ;

that is, the exterior angle  $ACD$  is greater than either  $CAB$  or  $ABC$ \*.

Q. E. D.

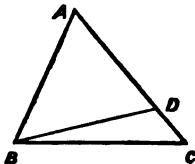
#### THEOREM 10.

*The greater side of every triangle has the greater angle opposite to it.*

\* Euclid, I. 16.



*Part. En.* Let  $ABC$  be a triangle having  $AC$  greater than  $AB$ ;



it is required to prove that the angle  $ABC$  is greater than the angle  $ACB$ .

*Proof.* From  $AC$  cut off  $AD = AB$ ; and join  $DB$ .

Because  $AD = AB$ ; (Constr.)

therefore the angle  $ABD =$  the angle  $ADB$ . (Th. 6.)

But because  $ADB$  is the exterior angle of the triangle  $BDC$ ,

therefore the angle  $ADB$  is greater than the angle  $ACB$ ; (Th. 9.)

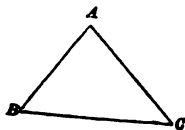
therefore also the angle  $ABD$  is greater than the angle  $ACB$ : much more then is the angle  $ABC$  greater than the angle  $ACB^*$ . Q. E. D.

#### THEOREM II.

*The greater angle of every triangle has the greater side opposite to it.*

*Part. En.* Let  $ABC$  be a triangle in which the angle  $B$  is greater than the angle  $C$ ;

it is required to prove that the side  $AC$  is greater than the side  $AB$ .



*Proof.* For  $AC$  must be either equal to  $AB$ , or less than  $AB$ , or greater than  $AB$ .

\* Euclid, I. 18.

But  $AC$  is not equal to  $AB$ , for then the angle  $B$  would be equal to the angle  $C$ . (Th. 6.)

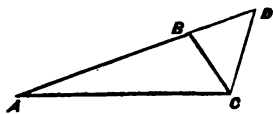
Nor is  $AC$  less than  $AB$ ,  
for then the angle  $B$  would be less than the angle  $C$ . (Th. 10.)

Therefore  $AC$  is greater than  $AB^*$ . Q. E. D.

## THEOREM 12.

*Any two sides of a triangle are together greater than the third side.*

*Part. En.* Let  $ABC$  be a triangle; it is required to prove that  $AB$  and  $BC$  are together greater than  $AC$ .



*Proof.* Produce  $AB$  to  $D$ , making  $BD = BC$ ;  
join  $DC$ .

Then because  $BD = BC$ , (Constr.)  
therefore the angle  $BCD =$  the angle  $BDC$ . (Th. 6.)

But the angle  $ACD$  is greater than the angle  $BCD$ ;  
therefore the angle  $ACD$  is greater than the angle  $BDC$ ;  
and therefore  $AD$  is greater than  $AC$ . (Th. 11.)

But  $AD$  is equal to  $AB$  and  $BC$  together,  
therefore  $AB$  and  $BC$  are together greater than  $AC^\dagger$ . Q. E. D.

*COR.* The difference of any two sides of a triangle is less than the third side.

\* Euclid, I. 19.

† Euclid, I. 20.

## THEOREM 13.

*If from the ends of the side of a triangle two straight lines be drawn to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.*

*Part. En.* Let  $ACB$  be a triangle, and from the ends of the side  $AB$  let two straight lines  $AP$ ,  $BP$  be drawn to a point  $P$  within the triangle; it is required to prove that  $AP$  and  $PB$  are less than  $AC$  and  $CB$ , but the angle  $APB$  greater than the angle  $ACB$ .

*Proof.* Produce  $AP$  to meet  $BC$  in  $Q$ .

Because any two sides of a triangle are together greater than the third side,

(Th. 12.)

therefore  $AC$  and  $CQ$  are greater than  $AQ$ ;

add to each  $QB$ ;

therefore  $AC$  and  $CB$  are greater than  $AQ$  and  $QB$ .

Again, because  $PQ$  and  $QB$  are greater than  $PB$ ;

(Th. 12.)

add to each  $AP$ ;

therefore  $AQ$  and  $QB$  are greater than  $AP$  and  $PB$ ;

but  $AC$  and  $CB$  are greater than  $AQ$  and  $QB$ ;

much more then are  $AC$  and  $CB$  greater than  $AP$  and  $PB$ .

Again, because  $APB$  is the exterior angle of the triangle  $PQB$ :

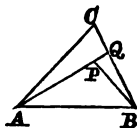
therefore the angle  $APB$  is greater than the angle  $PQB$ ;

(Th. 9.)

and because  $PQB$  is the exterior angle of the triangle  $ACQ$ ;

therefore the angle  $PQB$  is greater than the angle  $ACQ$ ;

(Th. 9.)

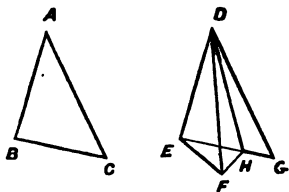


but the angle  $APB$  is greater than the angle  $PQB$ ;  
 much more then is the angle  $APB$  greater than the angle  
 $ACB^*$ . Q. E. D.

## THEOREM 14.

*If two triangles have two sides of the one equal to two sides of the other, each to each, but the included angles unequal, their bases are unequal, the base of that which has the greater angle being greater than the base of the other †.*

*Part. En.* Let  $ABC$ ,  $DEF$  be two triangles, having  
 $AB = DE$ ,  
 $AC = DF$ ,  
 but the included angle  $BAC$  greater than the included  
 angle  $EDF$ ;



it is required to prove that the base  $BC$  is greater than the base  $EF$ .

*Proof.* Place the point  $A$  on  $D$ , and  $AB$  along  $DE$ ;  
 then because  $AB = DE$ , (Hyp.)  
 therefore the point  $B$  will fall on the point  $E$ ;  
 and because the angle  $BAC$  is greater than the angle  $EDF$ ,  
 the line  $AC$  will fall outside  $DF$ , as  $DG$ ; (Hyp.)  
 and  $BC$  will fall as  $EG$ .

Let  $DH$  be the bisector of the angle  $FDG$ , (Ax. 4.)  
 meeting  $EG$  in  $H$ .

\* Euclid, I. 21.    † Euclid, I. 24.

Join  $FH$ .

Then because in the triangles  $FDH$ ,  $GDH$ , we have

$$FD = GD, \quad (\text{Hyp.})$$

$$DH \text{ common,}$$

$$\left. \begin{array}{l} \text{and the included angle } FDH = \text{the included angle } GDH, \\ \text{(Constr.)} \end{array} \right\}$$

$$\text{therefore } HF = HG; \quad (\text{Th. 5.})$$

and therefore  $EH$  and  $HF$  together are equal to  $EG$ .

But  $EH$  and  $HF$  together are greater than  $EF$ ; (Th. 12.)

therefore  $EG$  or  $BC$  is greater than  $EF$ . Q. E. D.

#### THEOREM 15.

*If two triangles have the three sides of the one equal to the three sides of the other, each to each, then the triangles are identically equal, and of the angles those are equal which are opposite to equal sides\*.*

*Part. En.* Let  $ABC$ ,  $DEF$  be two triangles which have

$$AB = DE,$$

$$BC = EF,$$

$$CA = FD,$$

then shall the triangles be equal in all respects.

*Proof.* The angle  $BAC$  must be either equal to  $EDF$ , or greater than  $EDF$ , or less than  $EDF$ .

But  $BAC$  is not greater than  $EDF$ ;

for then the base  $BC$  would be greater than the base  $EF$ .

(Th. 14.)

Nor is  $BAC$  less than  $EDF$ ;

for then the base  $BC$  would be less than the base  $EF$ :

(Th. 14.)

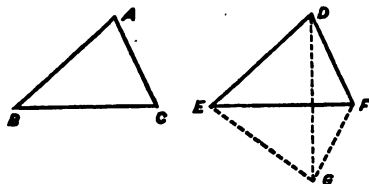
therefore the angle  $BAC$  is equal to the angle  $EDF$ .

Q. E. D.

\* Eucl. I. 8.

*Alternative Proof.*

If the point  $B$  were placed on  $E$ , and  $BC$  were placed along  $EF$ , then because  $BC = EF$  (Hyp.), therefore the



point  $C$  would fall on  $F$ : and let the triangles  $BAC$ ,  $EDF$  fall on opposite sides of  $EF$ ;  $BA$ ,  $AC$  falling as  $EG$ ,  $GF$ , and the angle  $BAC$  as  $EGF$ . Join  $DG$ .

Then because  $EG = ED$ , (Hyp.)

therefore the angle  $EDG =$  the angle  $EGD$ : (Th. 6.)

and because  $FG = FD$ , (Hyp.)

therefore the angle  $FDG =$  the angle  $FGD$ : (Th. 6.)

therefore the whole angle  $EDF =$  the whole angle  $EGF$ ;

but  $EGF = BAC$ ; (Constr.)

therefore  $EDF = BAC$ ;

and therefore the triangles  $BAC$ ,  $EDF$  are equal in all respects. (Th. 5.) Q. E. D.

NOTE. The student should examine for himself the cases in which  $DG$  passes through an extremity of the base, and passes outside the base.

## THEOREM 16.

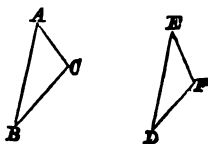
*If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one is greater than the base of the other, then the angle contained by the sides of that which has the greater base is greater than the angle contained by the sides of the other.*

*Part. En.* Let  $BAC$ ,  $DEF$  be two triangles which have

$$BA = DE,$$

$$AC = EF,$$

but the base  $BC$  greater than the base  $DF$ ;



it is required to prove that the angle  $BAC$  is greater than the angle  $DEF$ .

*Proof.* For the angle  $BAC$  must either be equal to the angle  $DEF$ , or less than the angle  $DEF$ , or greater than the angle  $DEF$ .

But the angle  $BAC$  is not equal to the angle  $DEF$ , for then the base  $BC$  would be equal to the base  $DF$ ; but it is not. (Th. 5.)

Nor is the angle  $BAC$  less than the angle  $DEF$ , for then the base  $BC$  would be less than the base  $DF$ ; but it is not. (Th. 14.)

Therefore the angle  $BAC$  must be greater than the angle  $DEF$ \*. Q. E. D.

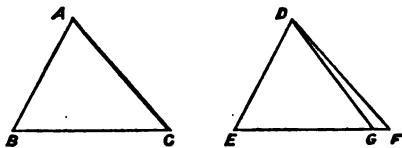
#### THEOREM 17.

*If two triangles have two angles of the one equal to two angles of the other, each to each, and have the sides opposite to one of the equal angles in each equal, then the triangles are equal in all respects, those sides being equal which are opposite to the equal angles.*

\* Euclid I. 25.

*Part. En.* Let the two triangles  $ABC$ ,  $DEF$  have the two angles

$$\left. \begin{array}{l} ABC = DEF, \\ ACB = DFE, \end{array} \right\} \text{ and the side } AB = \text{the side } DE;$$



it is required to prove the triangles are equal in all respects.

*Proof.* Let the point  $A$  be placed on the point  $D$ , and  $AB$  along  $DE$ ,

then because  $AB = DE$ , (Hyp.)

therefore the point  $B$  will fall on the point  $E$ .

And because the angle  $ABC$  is equal to the angle  $DEF$ , (Hyp.)

therefore the line  $BC$  will lie along  $EF$ .

And the point  $C$  will fall on  $F$ , for if it fell otherwise as  $G$ , then, since the angle  $ACB$  is equal to the angle  $EFD$ , (Hyp.) the angle  $EGD$  would be equal to the angle  $EFD$ , the exterior angle equal to the interior and opposite, which is impossible; (Th. 9.)

therefore the triangles would coincide and are equal in all respects,  $AC$  being equal to  $DF$ ,  $BC$  to  $EF$ , and the angle  $BAC$  to the angle  $EDF$ \*.

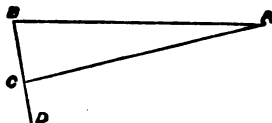
Q. E. D.

\* Eucl. I. 26. Part 2.



## THEOREM 18.

*Any two angles of a triangle are together less than two right angles.*



*Part. En.* Let  $ABC$  be a triangle, it is required to prove that any two of its angles  $ABC$  and  $ACB$  are together less than two right angles.

*Proof.* Produce the side  $BC$  to  $D$ .

Then because the exterior angle  $ACD$  is greater than the interior and opposite angle  $ABC$ ; (Th. 9.)  
add to each the angle  $ACB$ ;  
therefore the two angles  $ACD$  and  $ACB$  are greater than the two  $ABC$  and  $ACB$ .

But  $ACD$  and  $ACB$  are together equal to two right angles; (Th. 2.)  
therefore  $ABC$  and  $ACB$  are together less than two right angles\*.  
Q. E. D.

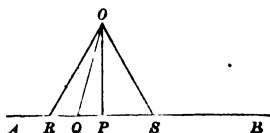
**COR. 1.** *If one angle of a triangle is right or obtuse, the others are acute.*

**COR. 2.** *From a given point outside a given straight line, only one perpendicular can be drawn to that straight line.*

\* Euclid, I. 17.

## THEOREM 19.

*Of all the straight lines that can be drawn from a given point to meet a given straight line, the perpendicular is the shortest; and of the others, those making equal angles with the perpendicular are equal; and that which makes a greater angle with the perpendicular is greater than that which makes a less.*



*Part. En.* Let  $O$  be the given point, and  $AB$  the given straight line, and let  $OP$  be the perpendicular,  $OQ$  an oblique;

it is required to prove first that  $OP$  is less than  $OQ$ .

*Proof.* Since any two angles of a triangle are together less than two right angles; (Th. 18.)

therefore  $OPQ$  and  $OQP$  are together less than two right angles:

but  $OPQ$  is a right angle; (Hyp.)

therefore  $OQP$  is less than a right angle.

And in the triangle  $OQP$ , since the angle  $OPQ$  is greater than the angle  $OQP$ ;

therefore  $OQ$  is greater than  $OP$ . (Th. 11.)

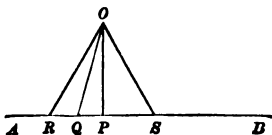
Again, let  $OS$ ,  $OR$  be obliques making equal angles with the perpendicular  $OP$ ;

it is required to prove that  $OR = OS$ .

Because in the triangles  $POR$ ,  $POS$

w.

the angle  $OPR = OPS$ , being right angles, } (Hyp.)  
 and the angle  $POR = POS$ , } (Hyp.)  
 and  $PO$  is common ;  
 therefore the triangles are equal in all respects, (Th. 7.)  
 and therefore  $OR = OS$ .



Lastly, let  $OR$  make a greater angle with the perpendicular than  $OQ$ ;  
 it is required to prove that  $OR$  is greater than  $OQ$ .

Because  $OQR$  is the exterior angle of the triangle  $OQP$ ;  
 therefore  $OQR$  is greater than  $OPQ$ ; (Th. 9.)  
 but  $OPQ$  is a right angle; (Hyp.)  
 therefore  $OQR$  is an obtuse angle;  
 therefore  $ORQ$  is an acute angle, and less than  $OQR$ ;  
 (Th. 18.)  
 and therefore  $OR$  is greater than  $OQ$ . Q. E. D.

*COR. Not more than two equal straight lines can be drawn from a given point to a given straight line.*

#### THEOREM 20.

*If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles opposite to two equal sides equal, the angles opposite to the other two equal sides are either equal or supplementary, and in the former case the triangles are equal in all respects.*

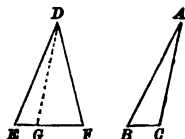
*Part. En.* Let  $ABC$ ,  $DEF$  be the two triangles, having the sides  $BA$ ,  $AC$  equal to the sides  $ED$ ,  $DF$  respectively, and having also the angle  $B =$  the angle  $E$ ;

it is required to prove that the angle  $C$  is equal or supplementary to the angle  $F$ ; and that when the angle  $C$  is equal to the angle  $F$ , the triangles are equal in all respects.

Fig. 1.



Fig. 2.



*Proof.* The contained angle  $A$  must be either equal or unequal to the contained angle  $D$ .

If  $A = D$ , as in Fig. 1, then, by Theorem 5, the triangles are equal in all respects; and the angle  $C$  is *equal* to the angle  $F$ .

If  $A$  is not equal to  $D$ , as in Fig. 2, let the point  $A$  be placed on  $D$ , and  $AB$  along  $DE$ ,

then, because  $AB = DE$ , (Hyp.)

the point  $B$  will coincide with the point  $E$ ;

and because the angle at  $B =$  the angle at  $E$ , (Hyp.)

therefore the line  $BC$  will lie along  $EF$ ,

and the point  $C$  will fall on  $EF$  as  $G$  :

and because  $AC = DF$ , (Hyp.)

therefore  $DG = DF$ ,

and therefore the angle  $DFE =$  the angle  $DGF$ ; (Th. 6.)

but  $DGF$  is supplementary to  $DGE$ , that is, to  $ACB$ , and therefore the angle  $F$  is *supplementary* to the angle  $C$ .

Q. E. D.

COR. *Hence the triangles are equal in all respects—*

(1) *If the two angles given equal are right angles or obtuse angles.*

For then the remaining angles must be acute, and therefore cannot be supplementary, and must therefore be equal by the Theorem, and therefore the triangles must be equal in all respects.

(2) *If the angles opposite to the other two equal sides are both acute, or both obtuse, or if one of them is a right angle.*

(3) *If the side opposite the given angle in each triangle is not less than the other given side.*

For then the given angles must be the greater of the two, and therefore the remaining angles must be both acute, and therefore cannot be supplementary, and must therefore be equal, by the Theorem, and therefore the triangles must be equal in all respects.

### EXERCISES ON THEOREMS OF EQUALITY.

The general method to be adopted in the solution of theorems of equality is the following. Examine fully the statement of the question; see what is included among the *data*: what lines and angles are *given* equal by *hypothesis*. Then see what is required to be proved, what lines or angles have to be proved to be equal. It may follow from the properties proved of a single triangle; or it may depend on the equality of a pair of triangles. In the latter case examine the triangles of which they form corresponding parts, and see whether the data are sufficient to prove *these* triangles equal. If the data are sufficient, the solution is effected by comparing the triangles, and shewing the required equality of the lines and angles; if not, the data must be used to establish results, which in their turn can be used to establish the conclusion required.

The beginner will do well to arrange his proofs in the manner shewn in the example, giving references in the margin.

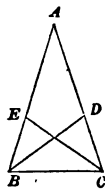
## EXERCISES.

Ex. (1). *The lines which bisect the angles at the base of an isosceles triangle, and meet the opposite sides, are equal.*

Let  $ABC$  be an isosceles triangle.

*Data.*  $AB=AC$ , and the angles at  $B$  and  $C$  bisected by  $BD$ ,  $CE$ .

*Proof.* In the triangles  $ACE$ ,  $ABD$  we have  
 $AC=AB$ , (Hyp.)  
 angle at  $A$  common,  
 and angle  $ACE=$ angle  $ABD$ . (Hyp. and Th. 6.)



Therefore the base  $CE=$ the base  $BD$ .

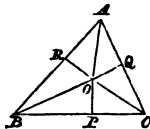
(Th. 7.) Q. E. D.

Ex. (2). *The bisectors of the three angles of a triangle will meet in one point.*

Let  $ABC$  be a triangle, and let the bisectors of the angles  $ABC$ ,  $ACB$  be  $BO$ ,  $CO$ , meeting in  $O$ ; then the Theorem will be proved if we can shew that  $AO$  is the bisector of the angle  $BAC$ .

Let perpendiculars  $OP$ ,  $OQ$ ,  $OR$  be drawn to the three sides  $BC$ ,  $CA$ ,  $AB$ .

*Proof.* In the triangles  $OQC$ ,  $OPC$  we have  
 $OQC=OPC$ , (Constr.)  
 $OCQ=OCP$ , (Hyp.)  
 $OC$  common.



Therefore  $OQ=OP$  by Theorem 17.

Similarly from the triangles  $OPB$ ,  $ORB$ , it follows that  $OP=OR$ ; therefore  $OR=OQ$ ;

and therefore the right-angled triangles  $OQA$ ,  $ORA$  have the hypotenuse and one side of the one equal to the hypotenuse and one side of the other, and are therefore equal in all respects by Theorem 20, Cor. 1.

Therefore the angle  $OAQ=$ the angle  $OAR$ , that is,  $OA$  is the bisector of the angle  $BAC$ .

## EXERCISES FOR SOLUTION.

1.  $OA$  and  $OB$  are any two equal lines, and  $AB$  is joined; shew that  $AB$  makes equal angles with  $OA$  and  $OB$ .

2. If the bisectors of the equal angles  $B$ ,  $C$  of an isosceles triangle meet in  $O$ , shew that  $OBC$  is also an isosceles triangle.

3. The line drawn to bisect the vertical angle of an isosceles triangle will also bisect the base, and be perpendicular to it.

4. The lines joining the middle points of the sides of an isosceles triangle to the opposite extremities of the base will be equal to one another.

5. The line drawn from the vertex of an isosceles triangle to bisect the base will cut it at right angles, and bisect the vertical angle.

6. Prove that the lines which bisect the sides of a triangle and are perpendicular to them meet in one point.

7. The perpendiculars let fall from the extremities of the base of an isosceles triangle upon the opposite sides will be equal, and will make equal angles with the base.

8. The perpendicular let fall from the vertex of an isosceles triangle to the base, will bisect the base and the vertical angle.

9. If two exterior angles of a triangle be bisected by straight lines which meet in  $O$ , prove that the perpendiculars from  $O$  on the sides or sides produced of the triangle are equal to one another.

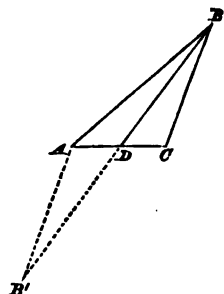
## EXERCISES ON THEOREMS OF INEQUALITY.

*The line that joins the vertex to the middle point of the base of a triangle is less than half the sum of the two sides.*

Let  $D$  be the middle point of  $AC$ , then is  $BD$  less than half the sum of  $AB, BC$ .

*Proof.* Produce  $BD$  to  $B'$ , making  $DB' = DB$ . Join  $AB'$ .

Then since the two triangles  $BDC, B'DA$  have two sides  $BD, DC$  and the included angle  $BDC$  of the one respectively equal to the two sides  $B'D, DA$  and the included angle  $B'DA$  of the other, therefore (Theorem 5) the base  $BC =$  the base  $AB'$ ;



but

$$B'A + AB > B'B,$$

(Th. 12.)

$$\therefore AB + BC > B'B, \text{ which is twice } BD,$$

that is,  $BD$  is less than half the sum of  $BC$  and  $BA$ .

## EXERCISES FOR SOLUTION.

1. Prove that any one side of a four-sided figure is less than the sum of the other three sides.
2. Prove that the sum of the lines which join the opposite angles of any four-sided figure is together greater than the sum of either pair of opposite sides of the figure.
3. Prove that the sum of the diagonals of a quadrilateral figure is less than the sum of the four lines which can be drawn to the angles from any other point than the intersection of the diagonals.



4.  $O$  is any point within the triangle  $ABC$ ; prove that  $OA + OB + OC$  are less than the sum and greater than half the sum of  $AB + BC + CA$ .

5. Prove that the sum of the four sides of a quadrilateral figure is greater than the sum and less than twice the sum of the diagonals.

6. If  $ABC$  is a triangle in which  $AB$  is greater than  $AC$ , and  $D$  is the middle point of  $BC$ , and  $AD$  is joined, prove that the angle  $ADB$  is an obtuse angle.

7. Prove that the sum of the three sides of a triangle is greater than the sum of the three medians.

NOTE. *The median of a triangle is the line that joins any angle to the middle point of the opposite side.*

8. Prove that the sum of the three medians of a triangle is greater than half the sum of the sides.

## QUESTIONS ON SECTION II.

1. Give the meaning and derivation of the words *triangle*, *perimeter*, *isosceles*, *equilateral*, *hypotenuse*, *median*.

2. If a triangle is isosceles, the angles at its base will be equal. Enunciate the obverse, converse and contrapositive theorems.

3. Apply Theorem 7 to find the height of a tower.

4. Prove Theorem 6 in the manner of Theorem 8.

5. Why cannot Theorem 15 be proved in the same manner as Theorem 5?

6. Prove that only one perpendicular can be drawn from a given point to a given straight line.

7. Prove fully the corollary to Theorem 19.

8. Enumerate the five cases in which the equality of three parts in a pair of triangles involves the equality in all respects.

9. Mention cases, and draw the figures, in which two triangles are equal in three respects but not in all.

10. Prove fully the corollaries to Theorem 20.

11. Prove Theorem 19 by conceiving the figure to be folded down over the line  $AB$ ,  $O$  falling on a point  $O'$ , and  $RO$ ,  $QO$ ,  $PO$ , on  $RO'$ ,  $QO'$ ,  $PO'$ , and using Theorems 12 and 13.

12. In Theorem 9, prove fully that  $ACD$  is greater than  $ABC$ .

13. Why is it necessary, in the enunciation of Theorem 9, to say interior *and opposite* angles?

14. What is the magnitude indirectly measured in Theorem 9?

15. Enunciate Theorem 10 formally. Is it merely the obverse of Theorem 6, or does it contain an additional geometrical fact?

16. Prove Theorem 10 by reversal and superposition, using Theorem 9.

17. Shew how Theorem 12 depends ultimately on the Axioms.

18. Which Theorem in this Section proves that as you increase the angle between the legs of a pair of compasses you also increase the distance between their points?

19. Shew the relation of Theorems 14, 15 and 16 to Theorem 5.

20. Enunciate the contra-positives of Theorems 9 and 18.

## SECTION III.

## PARALLELS AND PARALLELOGRAMS.

*Def. 35.* *Parallel* straight lines are such as are in the same plane and being produced to any length both ways do not meet.

*Axiom 5.* Two straight lines that intersect one another cannot both be parallel to the same straight line.

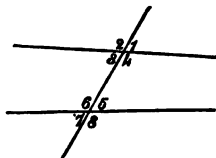
*Def. 36.* A *trapezium* is a quadrilateral that has only one pair of opposite sides parallel.

This figure is sometimes called a *trapezoid*.

*Def. 37.* A *parallelogram* is a quadrilateral whose opposite sides are parallel.

*Def. 38.* When a straight line intersects two other straight lines it makes with them eight angles, which have received special names in relation to one another.

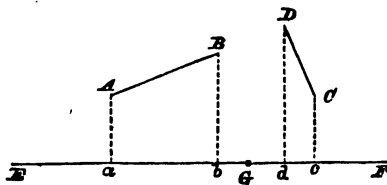
Thus in the figure 1, 2, 7, 8 are called *exterior* angles, and 3, 4, 5, 6, *interior* angles; again, 4 and 6, 3 and 5, are called *alternate* angles; lastly, 1 and 5, 2 and 6, 3 and 7, 4 and 8, are called *corresponding* angles.



*Def. 39.* The *orthogonal projection* of one straight line on another straight line is the portion of the latter intercepted

between perpendiculars let fall on it from the extremities of the former.

Thus the projections of  $AB$ ,  $CD$  on  $EF$  are the lines  $ab$ ,  $cd$  respectively.

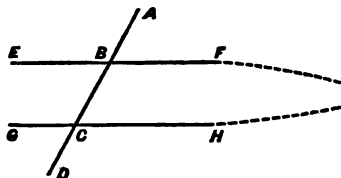


It is clear that the line  $EF$  must be supposed indefinitely long. There could be no projection of  $AB$  on the terminated line  $GF$ .

#### THEOREM 21.

*If one straight line intersects two other straight lines so as to make the alternate angles equal, the straight lines are parallel.*

*Part. En.* Let  $ABCD$  intersect  $EF$  and  $GH$ , and make the angle  $EBC$  equal to its alternate angle  $BCH$ ; it is required to prove that  $EF$  is parallel to  $GH$ .



*Proof.* For if  $EF$  and  $GH$  meet towards  $F$ ,  $H$ , they would form a triangle with  $BC$ ;

and  $EBC$  would be its exterior angle, and therefore greater than the interior and opposite angle  $BCH$ . (Th. 9.)

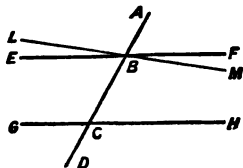
But  $EBC$  is equal to  $BCH$ , (Hyp.)  
therefore  $EF$  and  $GH$  do not meet towards  $F, H$ .

Similarly they do not meet towards  $E, G$ ;  
that is,  $EF$  is parallel to  $GF^*$ . (Def. 35.) Q. E. D.

### THEOREM 22.

*If two straight lines are parallel, and are intersected by a third straight line, the alternate angles are equal†.*

*Part. En.* Let  $EF$  and  $GH$  be parallel straight lines, and let  $ABCD$  intersect them;  
it is required to prove that the alternate angles  $EBC, BCH$  are equal.



*Proof.* For if  $EBC$  were not equal to  $BCH$ ,  
let some other line  $LBM$  be drawn through  $B$  making the angle  $LBC$  equal to the alternate angle  $BCH$ ;  
then  $LM$  would be parallel to  $GH$ . (Th. 21.)

But  $EF$  is parallel to  $GH$ ; (Hyp.)  
that is, two intersecting lines  $LM, EF$  would be both parallel to  $GH$ ; which is impossible. (Ax. 5.)

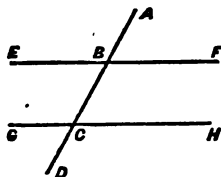
Therefore  $EBC$  is equal to  $BCH$ , that is, the alternate angles are equal. Q. E. D.

\* Euclid, I. 27.

† Euclid, I. 29.

## THEOREM 23.

*If a straight line intersects two other straight lines and makes either a pair of alternate angles equal, or a pair of corresponding angles equal, or a pair of interior angles on the same side supplementary; then, in each case, the two pairs of alternate angles are equal, and the four pairs of corresponding angles are equal, and the two pairs of interior angles on the same side are supplementary.*



*Part. En.* Let the straight line  $ABCD$  intersect the two straight lines  $EF$ ,  $GH$ , and make the alternate angles  $EBC$ ,  $BCH$  equal; then will the other alternate angles  $FBC$ ,  $BCG$  be equal, and the four pairs of corresponding angles be equal, and the two interior angles on the same side be supplementary.

Because  $EBC = BCH$ , (Hyp.)

and  $EBC = ABF$  being vertically opposite angles, (Th. 4.)

therefore  $ABF = BCH$ ;

therefore also their supplements, the angles  $ABE$  and  $BCG$  are equal.

Therefore also the angles which are respectively vertically opposite to these angles are equal,

that is, the angle  $CBF = DCH$ ,

and  $EBC = GCD$ .

Again, because the angle  $EBC$  = the alternate angle  $BCH$  add to each the angle  $CBF$ ;  
therefore the two angles  $EBC$ ,  $CBF$  are equal to the two  $CBF$ ,  $BCH$ ;  
but the two  $EBC$ ,  $CBF$  are together equal to two right angles;  
therefore the two  $CBF$ ,  $BCH$  are together equal to two right angles.

And in the same way it may be shewn that if two corresponding angles are given equal, or if two interior angles on the same side are supplementary, then the alternate angles will be equal.

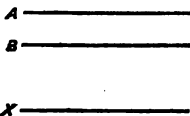
COR. Hence if two parallel straight lines are intersected by a third straight line, the corresponding angles are equal, and the interior angles on the same side are supplementary; and conversely.

#### THEOREM 24.

*Straight lines which are parallel to the same straight line are parallel to one another\*.*

*Part. En.* Let  $A$  and  $B$  be each of them parallel to  $X$ , it is required to prove that  $A$  is parallel to  $B$ .

*Proof.* If  $A$  intersected  $B$ , then two intersecting lines,



$A$ ,  $B$  would each be parallel to a third line  $X$ , which is impossible, by Axiom 5.

Therefore  $A$  does not intersect  $B$ ,  
that is,  $A$  is parallel to  $B$ .

Q. E. D.

\* Euclid, I. 30.

## REMARKS.

It must be observed that two parallels, and a straight line intersecting them, are a special case of the triangle, the vertex, or intersection of two of the lines, being removed to an infinite distance. In Th. 9 it was proved that the exterior angle of a *triangle* is *greater* than the interior and opposite angle: from which the contra-positive theorem (Th. 21) logically follows, that if the exterior angle is *equal* to the interior and opposite angle, the lines do *not* form a triangle.

Theorem 22 is in fact proved by the rule of identity (p. 4).

Since there is only one straight line through  $B$  that makes the alternate angles equal;

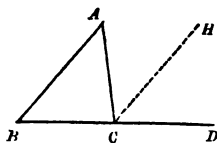
and only one straight line through  $B$  that is parallel to  $GH$ ; (Ax. 5.)

and the line that makes the alternate angles equal is the parallel; (Th. 21.)

therefore the parallel makes the alternate angles equal.

## THEOREM 25.

*If one side of a triangle be produced the exterior angle will be equal to the two interior and opposite angles, and the three interior angles of a triangle are together equal to two right angles.*



Let one side  $BC$  of the triangle  $ABC$  be produced to  $D$ : then shall the angle  $ACD$  = the sum of the angles  $ABC$ ,  $CAB$ ; and the three angles  $ABC$ ,  $BCA$ ,  $CAB$  shall be together equal to two right angles.

*Proof.* For if through  $C$  a line  $CH$  were drawn parallel to  $BA$ ,

the angle  $HCD$  = the corresponding angle  $ABC$ , (Th. 22.)

and the angle  $ACH$  = the alternate angle  $BAC$ ; (Th. 22.)

$\therefore$  the whole angle  $ACD$  = the two angles  $ABC + BAC$ .



Again, if  $ACB$  be added to these,  
the two angles  $ACD + ACB =$  the three angles  $ABC + BCA + CAB$ .

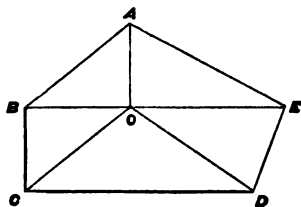
But  $ACD + ACB =$  two right angles; (Th. 4.)  
therefore  $ABC + BCA + CAB =$  two right angles\*. Q. E. D.

COR. In a right-angled triangle the two acute angles together make up one right angle.

### THEOREM 26.

*The interior angles of any polygon are together less than twice as many right angles as the figure has sides by four right angles.*

*Part. En.* Let  $ABCDE$  be any polygon; it is required to prove that its interior angles are together less than twice as many right angles as the figure has sides by four right angles.



*Proof.* Take any point  $O$  within the polygon; and join  $OA, OB, OC, OD, OE$ .

Then there are as many triangles having  $O$  as a common vertex as the figure has sides.

And, since the interior angles of a triangle are equal to two right angles; (Th. 25.)

\* Euclid, I. 32.

SECT. III.] PARALLELS AND PARALLELOGRAMS. 49

therefore all the angles of all the triangles are equal to twice as many right angles as the figure has sides.

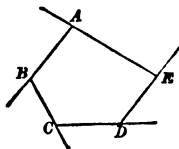
But all the angles of all the triangles make up all the angles of the polygon together with the angles at  $O$ , which are equal to four right angles; (Th. 4. Cor.)

therefore all the angles of the polygon, together with four right angles, are equal to twice as many right angles as the figure has sides;

that is, all the angles of the polygon are together less than twice as many right angles as the figure has sides by four right angles\*.

*COR. The exterior angles of any convex polygon are together equal to four right angles.*

Let  $ABCDE$  be a convex polygon having all its sides  $AB, BC, CD, DE, EA$  produced;



it is required to prove that the sum of its exterior angles is equal to four right angles.

*Proof.* Each interior angle together with its adjacent exterior angle are equal to two right angles; (Th. 2.)

therefore all the interior angles together with all the exterior angles are equal to twice as many right angles as the figure has sides;

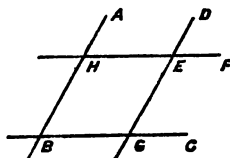
\* Euclid, I. 32. Cor. 1.

but all the interior angles, together with four right angles, are equal to twice as many right angles as the figure has sides; (Th. 26.)

therefore all the exterior angles are equal to four right angles\*. Q. E. D.

## THEOREM 27.

*The adjoining angles of a parallelogram are supplementary and the opposite angles are equal.*



*Part. En.* Let  $HBGE$  be a parallelogram; that is, let  $HE$ ,  $EG$  be respectively parallel to  $BG$ ,  $BH$ ; (Def. 37.)

it is required to prove that its adjoining angles  $EHB$ ,  $HBG$  are supplementary, and its opposite angles  $HBG$ ,  $HEG$  are equal.

*Proof.* Because  $HE$  is parallel to  $BG$ , (Hyp.) and  $HB$  meets them,

therefore  $HBG$  is supplementary to  $EHB$ . (Th. 23. Cor.)

And because  $HB$  is parallel to  $EG$ , (Hyp.) and  $HE$  meets them,

therefore  $HEG$  is supplementary to  $EHB$ ; (Th. 23. Cor.)

but  $HBG$  is also supplementary to  $EHB$ ,

therefore  $HEG$  is equal to  $HBG$ . (Th. 1. Cor. 3.) Q. E. D.

\* Euclid, I. 32. Cor. 2.

COR. 1. Hence if one of the angles of a parallelogram is a right angle, all its angles are right angles.

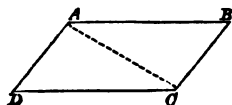
COR. 2. If two straight lines are respectively parallel to two other straight lines they will include equal angles towards the same parts.

Def. 40. A right-angled parallelogram is called a rectangle.

### THEOREM 28.

The opposite sides of a parallelogram are equal to one another, and the diagonal bisects it.

Part. En. Let  $ABCD$  be a parallelogram, that is, let  $AB$  be parallel to  $CD$ , and  $AD$  to  $BC$ ; it is required to prove that  $AB$  is equal to  $DC$ , and  $AD$  to  $BC$ .



Proof. Join  $AC$ .

Then because  $AB$  is parallel to  $DC$ , and  $AC$  meets them; (Hyp.)  
therefore the angle  $BAC$  is equal to the alternate angle  $ACD$ . (Th. 22.)

And because  $AD$  is parallel to  $BC$ ; (Hyp.)  
therefore the angle  $BCA$  is equal to the alternate angle  $CAD$ : (Th. 22.)

therefore in the triangles  $BAC$ ,  $DCA$  we have

the angle $BAC$ = the angle $DCA$ ,	}
and the angle $BCA$ = the angle $DAC$ ;	
and the side $AC$ adjacent to the equal	
angles common;	

therefore the triangles are equal in all respects, (Th. 7.)

that is,  $AB$  is equal to  $DC$ ,  $AD$  to  $BC$ , and the area  $ABC$  to the area  $ADC$ .\*

Q. E. D.

*COR.* Hence if the adjacent sides of a parallelogram are equal, all its sides are equal.

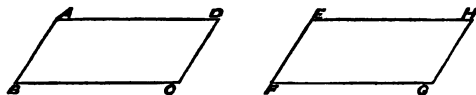
*Def. 41.* A parallelogram all whose sides are equal is called a *rhombus*.

*Def. 42.* A *square* is a rectangle that has all its sides equal.

### THEOREM 29.

*If two parallelograms have two adjacent sides of the one respectively equal to two adjacent sides of the other, and likewise an angle of the one equal to an angle of the other; the parallelograms are identically equal.*

*Part. En.* Let  $ABCD$ ,  $EFGH$  be two parallelograms which have two adjoining sides  $AB$ ,  $BC$  of the one equal respectively to two adjoining sides  $EF$ ,  $FG$  of the other, and have likewise the included angles  $B$  and  $F$  equal;



it is required to prove that the parallelograms are identically equal.

*Proof.* For if the point  $B$  were placed on  $F$ , and the line  $BC$  along the line  $FG$ ;

then because  $BC = FG$ , (Hyp.)

therefore the point  $C$  will fall on  $G$ ;

and because the angle  $ABC =$  the angle  $EFG$ , (Hyp.)

\* Euclid, I. 34.

therefore  $BA$  will fall along  $FE$ ;

and because  $BA = FE$ , (Hyp.)

therefore the point  $A$  will fall on  $E$ .

And because  $AD$  is parallel to  $BC$ , (Hyp.)

therefore  $AD$  will fall along  $EH$ ; (Ax. 5.)

and similarly  $CD$  will fall along  $GH$ ,

and therefore the point  $D$  will fall on the point  $H$ ;

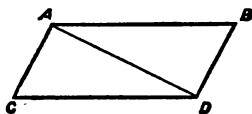
that is, the parallelograms are identically equal. Q. E. D.

**COR.** *Two rectangles are equal, if two adjacent sides of the one are respectively equal to two adjacent sides of the other; and two squares are equal, if a side of the one is equal to a side of the other.*

#### THEOREM 30.

*If a quadrilateral has two opposite sides equal and parallel, it is a parallelogram.*

**Part. En.** Let  $ABCD$  be a quadrilateral in which the opposite sides  $AB$ ,  $CD$  are equal and parallel;



it is required to prove that  $AC$  is equal and parallel to  $BD$ .

**Proof.** Join  $AD$ .

Then because  $AB$  is parallel to  $CD$ , (Hyp.)

therefore the angle  $BAD$  is equal to the alternate angle  $ADC$ ; (Th. 22.)

and therefore in the triangles  $BAD$ ,  $CDA$ , we have

$$\left. \begin{array}{l} BA = CD, \quad (\text{Hyp.}) \\ AD \text{ common,} \\ \text{and the contained angles } BAD, CDA \\ \text{equal;} \end{array} \right\}$$

therefore the triangles are equal in all respects; (Th. 5.)

that is,  $BD = AC$ ;

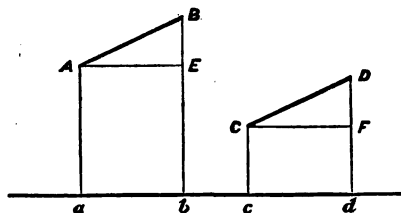
and the angle  $BDA =$  the angle  $CAD$ ;

but these are alternate angles;

and therefore  $AC$  is parallel to  $BD$ \*. (Th. 21.) Q. E. D.

### THEOREM 31.

*Straight lines which are equal and parallel have equal projections on any other straight line; conversely, parallel straight lines which have equal projections on another straight line are equal; and equal straight lines, which have equal projections on another straight line, are equally inclined to that line.*



*Part. En.* Let  $AB$ ,  $CD$  be equal and parallel straight lines, and let  $ab$ ,  $cd$  be their projections on any other straight line. Then shall  $ab$  be equal to  $cd$ .

*Proof.* Through  $A$ ,  $C$  draw  $AE$ ,  $CF$  parallel to  $abcd$ , meeting  $Bb$ ,  $Dd$  in  $E$ ,  $F$ .

\* Euclid, I. 33.

Then because  $BA$ ,  $AE$ ,  $BE$  are respectively parallel to  $DC$ ,  $CF$ ,  $DF$ , (Hyp.)

therefore the angle  $BAE =$  the angle  $DCF$ ,

and the angle  $BEA =$  the angle  $DFC$ , (Th. 27. Cor. 2.)

and the hypotenuse  $AB =$  the hypotenuse  $CD$ ; (Hyp.)

therefore  $AE = CF$ : (Th. 17.)

but  $AE = ab$ , and  $CF = cd$ , (Th. 28.)

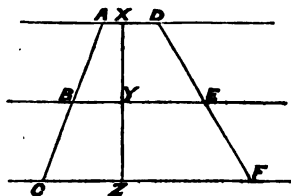
and therefore  $ab = cd$ . Q. E. D.

Similarly the converse propositions may be proved.

### THEOREM 32.

*If there are three parallel straight lines, and the intercepts made by them on any straight line that cuts them are equal, then the intercepts on any other straight line that cuts them are equal.*

Let the three parallel straight lines  $AD$ ,  $BE$ ,  $CF$  make equal intercepts on the straight line  $AC$ , that is, let  $AB = BC$ .



Then shall the intercepts on any other line  $DEF$  be equal, that is,  $DE$  shall be equal to  $EF$ .



If one of the straight lines is perpendicular to the parallels, what is required to be proved follows directly from Theorem 31.

But if neither of the straight lines is perpendicular to the parallels, draw any straight line  $XYZ$  perpendicular to the parallels.

Then by Th. 31, the equal straight lines  $AB$ ,  $BC$  have equal projections,

and therefore

$$XY = YZ.$$

And again by the same Theorem, because  $XY = YZ$ , and that these are the projections of  $DE$ ,  $EF$ ,

therefore

$$DE = EF.$$

Q. E. D.

**COR. 1.** *The straight line drawn through the middle point of one of the sides of a triangle parallel to the base passes through the middle point of the other side.*

**COR. 2.** *The straight line joining the middle points of two sides of a triangle is parallel to the base.*

#### EXERCISES FOR SOLUTION.

1. If  $ABC$  is an isosceles triangle and  $A$  is double of either  $B$  or  $C$ , shew that  $A$  is a right angle.
2. If  $ABC$  is an isosceles triangle and  $A$  is half of either  $B$  or  $C$ , shew that  $A$  is two-fifths of a right angle.
3. Find the angle between the lines that bisect the angles at the base of the triangle in the last question.

4. The perpendiculars let fall from the extremities of the base of an isosceles triangle on the opposite sides will include an angle supplementary to the vertical angle of the triangle.

5. Shew that the angles of an equiangular triangle are equal to two-thirds of a right angle.

6. Find the magnitude of the angle of a regular octagon.  
(Th. 26.)

7. How many equiangular triangles can be placed so as to have one common angular point, and fill up the space round it?

8. Shew that three regular hexagons can be placed so as to have a common point, and fill up the space round that point.

9. Shew that two regular octagons and one square have the same property.

Draw a pattern consisting of octagons and squares.

10. Shew that the angle of a regular pentagon is to the angle of a regular decagon as 3 to 4.

11. If a line is perpendicular to another it will be perpendicular to every line parallel to it.

12. If a polygon is equilateral, does it follow that it is equiangular, and conversely?

13. How many diagonals can be drawn in a pentagon? How many in a decagon? How many in a polygon of  $n$  sides.

14. Shew that a square, a hexagon and a dodecagon will fill up the space round a point; and make a pattern of these polygons.

15. Examine whether a square, a pentagon and an icosagon have the same property; and also whether a pattern can be constructed of pentagons and decagons.

16. The exterior angle of a regular polygon is one-third of a right angle: find the number of sides in the polygon.

17. Two lines intersecting in  $A$  are respectively perpendicular to two lines intersecting in  $B$ : prove that any angle at  $A$  is equal or supplementary to any angle at  $B$ .

18. Shew that a trapezium may be divided into a parallelogram and a triangle.

19. The diagonals of any parallelogram will bisect one another.

20. The diagonals of a rhombus will bisect one another at right angles.

21. If two straight lines be drawn bisecting one another, and their extremities be joined, the figure so formed will be a parallelogram.

22. Given that a four-sided figure has its opposite sides equal, prove that it must be a parallelogram.

23. Prove that the diagonals of a rectangle are equal to one another.

24. Shew that if one element (a side) is given, a square is determined; if two elements (a side and angle), a rhombus is determined; also that if two elements (two sides) are given, a rectangle is determined: and find the number of elements required to determine a parallelogram, a trapezium, a quadrilateral, a pentagon, and a polygon of any number ( $n$ ) of sides.

## QUESTIONS ON SECTION III.

1. Give the derivation of the words parallel, parallelogram, trapezium.
2. What is indirectly ascertained in Theorem 21? Would it be possible to ascertain it directly?
3. Prove Th. 24 by drawing a straight line to intersect  $A, B$  and  $X$ , and using Theorems 21, 22.
4. Given two angles of a triangle to be respectively  $72^{\circ}.15'.47''$  and  $83^{\circ}.51'.16''$ , find the third angle.
5. If one angle of a triangle is equal to the other two, prove that it must be a right angle.
6. Isosceles triangles having equal vertical angles must have equal base angles.

## SECTION IV.

## PROBLEMS.

IN the Science of Geometry there are not only theorems to be proved, but constructions to be effected, which are called *problems*. Geometers have always imposed certain limitations on themselves with respect to the instruments which might be used in these constructions. There is no reason why any convenient instrument used in the Art of Geometry, such as the square, parallel ruler, sector, protractor, should not be supposed to be used also in the Science; but the ruler and compasses suffice for nearly all the simpler constructions, and those which cannot be effected by their means are considered as not forming a part of Elementary Geometry. These instruments are therefore *postulated* or requested (vid. p. 4). There are some problems, that seem at first sight not very difficult, that cannot be solved by the use of these instruments. We can, for example, bisect an angle; but we cannot, in general, trisect it, that is, divide it into three equal parts, by any combination of ruler and compasses.

It may be observed that the ruler is simply a straight edge, not graduated, and the compasses are supposed to be transferable from one part of the figure to another, the distance between the points being unaltered.

The solution of a problem in Elementary Geometry as above defined consists

- (1) in indicating how the ruler and compasses are to be used in effecting the construction required ;
- (2) in proving that the construction so given is correct ;
- (3) in discussing the limitations, which sometimes exist, within which alone the solution is possible.

We shall give several examples of such problems, and then discuss the principles of the methods we have used.

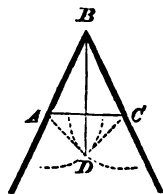
#### PROBLEM I.

*To bisect a given angle.*

*Construction.* Let  $ABC$  be the given angle.

Take any equal lengths  $BA$ ,  $BC$ , along its arms, and join  $AC$ .

With centre  $A$ , and any radius greater than half  $AC$ , describe a circle, and with centre  $C$ , and the same radius, describe another circle intersecting the former circle on the side of  $AC$  remote from  $B$  in  $D$ .



Join  $AD$ ,  $CD$ , and  $BD$ ;  
 $BD$  bisects the angle  $ABC$ .

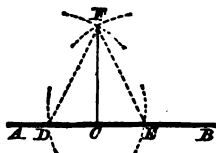
*Proof.* In the triangles  $ABD$ ,  $CBD$ ,  
 because  $AB = BC$ , (Constr.)  
 and  $BD$  is common,  
 and the base  $AD =$  the base  $DC$ , (Constr.)  
 therefore the angle  $ABD =$  the angle  $CBD$ , (Th. 15.)  
 that is,  $BD$  bisects the angle  $ABC$ .\*

\* Eucl. I. 9.

## PROBLEM 2.

*To draw a perpendicular to a given straight line from a given point given in it.*

*Construction.* With centre  $C$  and any radius describe a circle to cut the straight line in two points  $D, E$ , so that  $CD = CE$ .



With centre  $D$ , and any radius greater than  $DC$ , describe a circle, and with centre  $E$  and the same radius describe a circle, cutting the former in  $F$ .

Join  $FC$ ;

it is required to prove that  $FC$  is perpendicular to  $AB$ .

*Proof.* In the triangles  $DCF, ECF$ ,  
 because  $DC = CE$ , (Constr.)  
 $CF$  is common,  
 and the base  $DF =$  the base  $EF$ , (Constr.)  
 therefore the angle  $DCF =$  the angle  $ECF$ , (Th. 15.)  
 and therefore  $DCF$  and  $ECF$  are right angles\*. (Def. 14.)

NOTE.—This construction is usually effected in practice by means of the square.

It may be observed that this problem is only a special case of Prob. 1, the given angle being a straight angle.

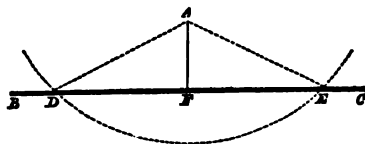
## PROBLEM 3.

*To draw a perpendicular to a given straight line from a given point outside it.*

Let  $BC$  be the given straight line,  $A$  the given point.

\* Euclid, I. 11.

*Construction.* With centre  $A$  describe a circle with any sufficient radius to cut  $BC$  in two points  $D, E$ .



Bisect the angle  $DAE$  by the line  $AF$ . (Prob. 1.)

Then  $AF$  shall be perpendicular to  $BC$ .

*Proof.* In the triangles  $AFD, AFE$ ,  
because  $AD = AE$ , (Constr.)  
and  $AF$  is common,

and the contained angle  $DAF =$  the contained angle  $EAF$ ;  
(Constr.)

therefore the angle  $AFD =$  the angle  $AFE$ ; (Th. 5.)

therefore  $AF$  is perpendicular to  $DE^*$ .

NOTE.—This construction also is usually effected in practice by means of the square.

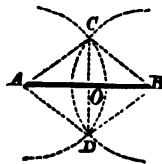
#### PROBLEM 4.

*To bisect a given straight line†.*

Let  $AB$  be the given straight line.

*Construction.* With centre  $A$  and any radius greater than half  $AB$  describe a circle, and with centre  $B$  and the same radius describe a circle intersecting the former in two points  $C$  and  $D$ .

Join  $CD$  cutting  $AB$  in  $O$ .



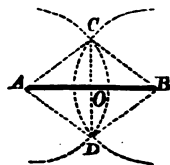
\* Euclid, I. 12.

† Euclid, I. 10.



Then  $O$  will be the 'point of bisection. Join  $AD$ ,  $DB$ .

*Proof.* Because  $AC = CB$ ; and  $CD$  is common to the two triangles  $ACB, DCB$ ; and the base  $AD$  is equal to the base  $DB$ ; therefore the angle  $ACD =$  the angle  $BCD$ ; therefore in the two triangles  $ACO, BCO$ , we have  $AC = BC$ , (Constr.)  $CO$  common, and the included angles  $ACO, BCO$  equal;



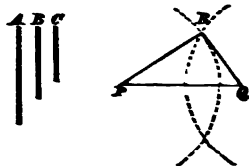
therefore the base  $AO =$  the base  $BO$ , (Th. 5.)  
or the line  $AB$  is bisected in  $O$ .

#### PROBLEM 5.

*To construct a triangle, having given the lengths of the three sides.*

Let the three given lengths be the lines  $A, B, C$ .

*Construction.* Draw a line  $PQ$  equal to one of them  $A$ . With centre  $P$  and radius equal to  $B$  describe a circle; and with centre  $Q$  and radius equal to  $C$  describe a circle. Let these circles intersect in  $R$ . Join  $RP, RQ$ .



$RPQ$  is the triangle required.

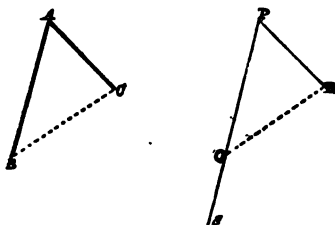
*Proof.* For  $RPQ$  has its three sides respectively equal to  $A, B$  and  $C$ .

*Limitation.*—It is necessary that any two of the lines  $A, B, C$  should be together greater than the third. For if  $B$  and  $C$  were together less than  $A$ , the circles in the figure would obviously not meet; and if they were together equal to  $A$ , the point  $R$  would be on  $PQ$ , and the triangle would become a straight line. Similarly if  $B$  were greater than  $A + C$  or  $C$  greater than  $A + B$ , the circles would not intersect. This limitation might be anticipated from the theorem before proved, that any two sides of a triangle are together greater than the third side, and is in fact its contrapositive.

## PROBLEM 6.

*At a given point in a given straight line to make an angle equal to a given angle.*

Let  $BAC$  be the given angle,  $P$  the given point in the line  $PQ$ .



*Constr.* Join any two points  $B, C$  in the arms of the given angle. Construct a triangle  $PQR$  having its three sides  $PQ, QR, RP$  respectively equal to  $AB, BC, CA$ .

(Prob. 5.)

*Proof.* In the triangles  $ABC, PQR$ ,

because

$$AB = PQ,$$

(Constr.)

$$AC = PR,$$

(Constr.)

and

$$BC = QR,$$

(Constr.)

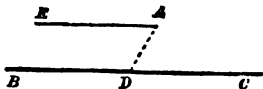
therefore the angle  $A =$  the angle  $P^*$ .

(Th. 12.)

\* Euclid, I. 23.

## PROBLEM 7.

*To draw through any point a straight line parallel to a given straight line.*

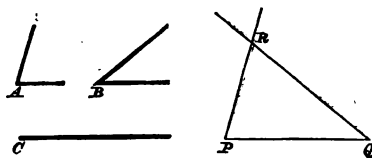


*Constr.* Let  $A$  be the given point,  $BC$  the given line. Draw any line  $AD$  to meet  $BC$ , and make the angle  $DAE$  equal to the alternate angle  $ADC$ . (Prob. 6.)

*Proof.* Because the alternate angles  $EAD$ ,  $ADC$  are equal, (Constr.) therefore  $AE$  is parallel to  $DE^*$ . (Th. 21.)

## PROBLEM 8.

*To construct a triangle, having given two angles and a side adjacent to both.*



Let  $A$ ,  $B$  be the two angles,  $C$  the given side.

Take a line  $PQ = C$ . At the points  $P$ ,  $Q$  make angles with  $PQ$  equal respectively to  $A$  and  $B$ . (Prob. 6.)

Let the lines which contain these angles meet in  $R$ .

Then  $RPQ$  is the triangle required.

*Proof.* For it has  $PQ = C$ , and the angles  $P$  and  $Q$  respectively equal to  $A$  and  $B$ .

\* Euclid, I. 31.

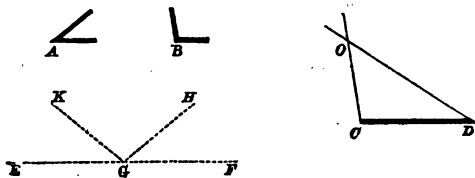
*Limitation.*—The two given angles must be together less than two right angles, or the lines  $PQ$ ,  $QR$  would not meet. This follows also from the theorem that any two interior angles of a triangle are together less than two right angles, and is the contrapositive of that theorem.

## PROBLEM 9.

*To construct a triangle, having given two angles and a side opposite to one of them.*

Let  $A$  and  $B$  be the given angles,  $CD$  the given side which is to be opposite to  $A$ .

*Construction.* Draw an indefinite straight line  $EF$ . At any point  $G$  in it make the angles  $FGH = A$ , and  $HGK = B$  (Prob. 6), then  $KGE$  will equal the third angle of the triangle, since the sum of the three angles of



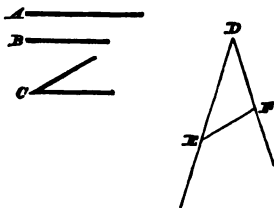
a triangle is equal to two right angles (Th. 25). At  $C$  and  $D$  make angles equal to  $HGK$  and  $KGE$ , and let their sides meet in  $O$ ; then  $OCD$  is the triangle required.

*Proof.* For  $OCD$  has  $CD$  equal to the given line, and the angles  $C$  and  $D$  equal respectively to the given angles.

*Limitation.*—As before, the two given angles must be together less than two right angles.

## PROBLEM 10.

To construct a triangle, having given two sides and the angle between them.



Let  $A$ ,  $B$  be the given sides,  $C$  the given angle.

*Construction.* Draw an angle  $D$  equal to the given angle, and take  $DE$ ,  $DF$  equal to  $A$  and  $B$ . Join  $EF$ .

*Proof.* For the triangle  $DEF$  has  $DE$ ,  $DF$  equal to the given lines  $A$  and  $B$ , and the included angle  $D$  equal to the given angle  $C$ .

*Remark.* In these problems we have found that one triangle and only one can be constructed to fulfil the conditions given. In other words, that with these *data* the triangle is *determinate*. Also we notice that in each case *three* elements in the triangle are *data* or given. We have given either the three sides, or two angles and the side adjacent to both, or two angles and a side opposite to one, or two sides and the included angle. And these cases correspond to the theorems proved above of the equality of triangles. For if *only one* triangle can be constructed so as to have its sides equal to three given lines, it is clear that if two triangles have the three sides of the one equal to the three sides of the other, these triangles must be identical, or be equal in all respects. And a similar remark may be made on the other cases we have considered.

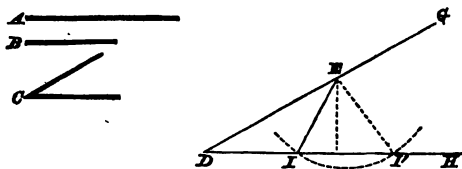
But there are cases in which the data may be insufficient to determine the triangle. For example, if only two sides are given, an

indefinite number of different triangles may be constructed to have these sides. Or if the three angles are given, their sum being equal to two right angles, an indefinite number of triangles may be constructed to have these three angles. And again it may be impossible to construct the triangle with the given data, as has been already shewn. In some cases moreover the solution is *ambiguous*, that is, there may be more than one triangle which fulfils the given conditions. The following is an important instance of this, and is usually called *the ambiguous case*, some consideration of which occurred in Theorem 20.

## PROBLEM II.

*To construct a triangle, having given two sides and an angle opposite to one of them.*

Let  $A, B$  be the given sides,  $C$  the angle to be opposite to the side  $B$ .



Take an angle  $GDH = C$ , take  $DE = A$ , and with centre  $E$  and radius  $= B$  describe a circle. If  $I$  is one of the points in which this circle meets the line  $DH$ , by joining  $EI$  we obtain a triangle which fulfils the given conditions.

But several cases may arise.

Let the given angle be acute, as in the figure.

Then, by Theorem 19,

(1) If  $B$  is less than the perpendicular from  $E$  on  $DH$ , the circle would not meet  $DH$ , and the triangle would be *impossible*.

(2) If  $B$  is equal to the perpendicular, the circle would meet  $DH$  at the foot of the perpendicular, and there would be *one triangle, right-angled*, which fulfils the given conditions.

(3) If  $B$  is greater than the perpendicular but less than  $DE$ , then the circle will meet  $DH$  in two points  $I, I'$  as in the figure, on the same side of  $D$ , and there will be *two triangles  $EDI, EDI'$*  which fulfil the given conditions.

(4) If  $B$  is equal to  $DE$ , the point  $I$  will coincide with  $D$ , and one of the two triangles disappears, and the other is isosceles.

(5) If  $B$  is greater than  $DE$ , the circle will meet  $DH$  in two points on the opposite sides of  $D$ , but one only of the triangles made by joining  $EI, EI'$  will be found to have the angle  $D$ , and the other will have the supplementary angle: that is, there will be only *one solution*.

The cases of the given angle being a right angle or an obtuse angle are left to the ingenuity of the student.

## SECTION V.

## LOCI.

WHEN a point has to be found to fulfil one given geometrical condition, the problem is indeterminate: that is, an infinite number of points can be found to fulfil the given condition. For example, if the problem is to find a point at a given distance from a given point, it is plain that all the points in the circumference of a circle, described with that point as centre and the given distance as radius, fulfil this condition. Or again, if a point has to be found at a given distance from a given straight line of indefinite length, it may lie anywhere on either of two straight lines parallel to the given line, and at the given distance from it on either side.

All the points which satisfy a single given geometrical condition lie in general in a line or lines: and this line, or these lines, are called the locus of the point under the given condition. Hence we get the following definition of a locus:

*Def.* If any and every point on a line or group of lines (straight or curved), and no other point, satisfies an assigned condition, that line or group of lines is called the *locus* of the point satisfying that condition.



In order that a line or group of lines  $A$  may be properly termed the locus of a point satisfying an assigned condition  $X$ , it is necessary and sufficient to demonstrate the two following associated Theorems :

*If a point is on  $A$ , it satisfies  $X$ .*

*If a point is not on  $A$ , it does not satisfy  $X$ .*

It may sometimes be more convenient to demonstrate the contrapositive of either of these Theorems.

The following examples of loci are important.

i. *The locus of a point at a given distance from a given point is the circumference of a circle having a radius equal to the given distance and having its centre at the given point.*

ii. *The locus of a point at a given distance from a given straight line is the pair of straight lines parallel to the given line, at the given distance from it, and on opposite sides of it.*

The proofs of these two Theorems are obvious.

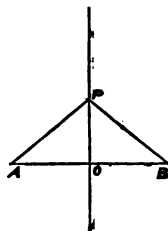
iii. *The locus of a point equidistant from two given points is the straight line that bisects, at right angles, the line joining the given points.*

*Part. En.* Let  $A$ ,  $B$  be the two given points ;  $P$  a point equidistant from  $A$  and  $B$ , so that  $PA = PB$  ;

it is required to find the locus of  $P$ .

*Constr.* Join  $AB$  ; and bisect it in  $O$ , and join  $PO$ .

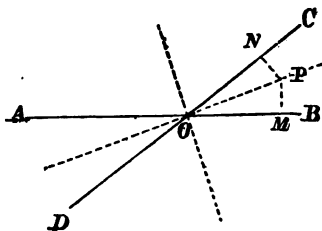
Then  $PO$  produced is the locus required.



*Proof.* In the triangles  $AOP$ ,  $BOP$ ,  
 because  $AO = OB$ , (Constr.)  
 and  $PO$  is common,  
 and  $AP = BP$ , (Hyp.)  
 therefore the angle  $AOP =$  the angle  $BOP$ , (I. 15.)  
 and therefore  $PO$  is at right angles to  $AB$ ;  
 that is, a point equidistant from  $A$  and  $B$  lies on the line  
 which bisects  $AB$  at right angles. Further, every point not  
 on the bisector, is at unequal distances from  $A$  and  $B$ , as  
 may be proved by Theorem 14, and therefore the line  
 which bisects  $AB$  at right angles is the locus of points equi-  
 distant from  $A$  and  $B$ .

iv. *The locus of a point equidistant from two intersecting straight lines is the pair of lines, at right angles to one another, which bisect the angles made by the given lines.*

Let  $AB$ ,  $DC$  intersect in  $O$ ; it is required to find a point equally distant from  $AB$  and  $DC$ .



Bisect the angle  $COB$ ; and in the bisector take any point  $P$ . Let fall  $PN$ ,  $PM$  perpendicular to  $DC$ ,  $AB$ .

In the triangles  $PON$ ,  $POM$ ,  
 because the angles  $PON$ ,  $PNO$  are respectively equal to  
 the angles  $POM$ ,  $PMO$ , (Constr.)

and the hypotenuse  $PO$  is common,  
therefore the triangles are equal in all respects,  
(Theorem 17).  
and therefore  $PN = PM$ .

In the same manner every point in the bisector of any one of the four angles at  $O$  is equally distant from  $AB$  and  $CD$ ;

that is, the locus of points equally distant from two straight lines which intersect, is the bisectors of the angles between the lines.

It may further be proved that no point not in a bisector is equally distant from these lines; that is, the bisectors are the *complete locus*.

#### EXERCISES.

Find the following loci :—

- (1) Of a point at a given distance from a given point.
- (2) Of a point at a given distance from a given line.
- (3) Of a point at a given distance from a given circle.
- (4) A horse is tethered by a chain fastened to a ring which slides on a rod bent into the form of a rectangle. Find the outline of the area over which he can graze.
- (5) Find the locus of a point equidistant from two given points. Prove that the locus found is *complete*.
- (6) Find the locus of points at which two equal lengths, adjacent or not adjacent, of a straight line subtend equal angles.

(7) Find part of the locus of points at which two adjacent sides of a square subtend equal angles.

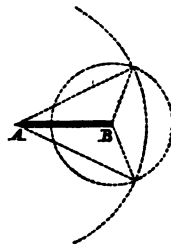
(8) Find the locus of a point at which two adjacent sides of a rectangle subtend supplementary angles.

### INTERSECTION OF LOCI.

When a point has to be found which satisfies *two* conditions, the problem is generally determinate if it is possible: and the method of loci is very frequently employed in discovering the point. For if the locus of points which satisfy each condition separately is constructed, it is obvious that the points which satisfy both conditions must be the points common to both loci, that is, must be the point or points where the loci intersect.

For example; a triangle is to be constructed on a given base with its sides of given lengths. Let  $AB$  be the base.

The two conditions are that the lengths of the two sides are given; the point sought for is the vertex: now the vertex must be at a certain distance from  $A$  = one of the given lengths; its locus is therefore a certain circle round  $A$  as centre. Similarly it must be at a certain distance from  $B$ ; its locus is therefore another circle round  $B$  as centre. The points of intersection of these circles are therefore the vertices of the two equal triangles which fulfil the given conditions.



It was this reasoning that suggested the construction in Problem 5.

Occasionally it will be found that with certain conditions among the data in the following Exercises the loci do not intersect, or the solution becomes impossible. So in the case given, it will not be difficult to see that the circles would not intersect unless any two of the given sides were greater than the third side. These conditions among the data for the possibility or impossibility of a solution should always be found.

The principle of the intersection of loci may be thus stated :

If  $A$  is the locus of a point satisfying the condition  $X$ , and  $B$  the locus of a point satisfying the condition  $Y$ ; then the intersections of  $A$  and  $B$ , and these points only, satisfy both the conditions  $X$  and  $Y$ .

The following examples of intersection of loci are important, and are at once demonstrated by the aid of the preceding examples of loci.

- i. *There is one and only one point in a plane which is equidistant from three given points not in the same straight line.*
- ii. *There are four and only four points in a plane each of which is equidistant from three given straight lines that intersect one another but not in the same point.*

#### EXERCISES ON INTERSECTION OF LOCI.

1. Find a point in a given straight line at equal distances from two given points. Construct the figures for all cases.
2. Find a point in a given straight line at a given distance from a given straight line.

3. Find a point in a given straight line at equal distances from two other straight lines.

4. On a given straight line to describe an equilateral triangle.

5. Describe an isosceles triangle on a given base, each of whose sides shall be double of the base.

6. Find a point at a given distance from a given point, and at the same distance from a given straight line.

7. Given base, sum of sides, and one of the angles at the base, construct the triangle.

8. Given base, difference of sides, and one of the angles at the base, construct the triangle.

9. Find a point at a given distance from the circumference of two given circles, the distances being measured along their radii or their radii produced.

10. A straight railway passes within a mile of a town. A place is described as four miles from the town, and half a mile from the railway. How many places satisfy the conditions?

11. Find a point equidistant from three given straight lines that intersect one another but not in the same point.

#### ANALYSIS AND SYNTHESIS.

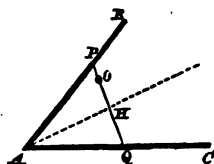
If problems cannot be solved by this method, it remains to attack them by the method, as it is called, of Analysis and Synthesis. This is not so much a method as a way of searching for a suggestion, and nothing but experience and ingenuity will here avail the student. The solution is

supposed to be effected, and relations among the parts of the figure are then traced until some relation is discovered which can give a clue to the construction. Nothing but seeing examples can make this clear.

(1) *It is required to draw a line to pass through a given point and make equal angles with two given intersecting lines.*

Let  $O$  be the given point,  $AB$ ,  $AC$  the given lines.

We reason as follows (*analysis*): suppose  $POQ$  were the line required, then the angle at  $P$  = angle at  $Q$ .



Therefore  $AP = AQ$ ; therefore if we bisected the angle  $A$ ,  $POQ$  would be at right angles to the bisector.

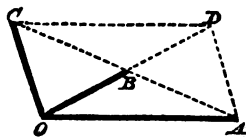
Now this is a suggestion we can work backwards from, and the construction is as follows.

*Synthesis.* Bisect the angle  $BAC$ , and let fall  $OH$  a perpendicular to the bisector, and let it meet the lines in  $P$ ,  $Q$ , and  $POQ$  can then be proved to be the line required.

(2) *It is required to draw from a given point three straight lines of given lengths, so that their extremities may be in the same straight line, and intercept equal distances on that line.*

*Analysis.* Suppose  $OA$ ,  $OB$ ,  $OC$  were the three lines, so that  $CBA$  is a straight line, and  $CB = BA$ .

Then it occurs to us that if



$OB$  were prolonged to  $D$ , making  $BD = OB$ , then  $CD$  and  $DA$  would be respectively parallel and equal to  $OA$  and  $OC$  (see § 3, Ex. 21); and that the sides of the triangle  $DOA$  are respectively equal to  $OA$ ,  $OC$  and  $2OB$ . Hence the construction is suggested.

*Synthesis.* Make a triangle  $DOA$  whose sides are  $OA$ ,  $OC$ , and  $2OB$ ; complete the figure, by drawing  $DC$ ,  $OC$  parallel to  $OA$ ,  $AD$ ; and the other diagonal  $ABC$  will be the line required. For it may be shewn that  $AB = BC$ .

The student must not be surprised if he finds problems of this class difficult. For there is nothing except previous knowledge of geometrical facts to point out which of the many relations of the parts of the figure are to be followed up in order to arrive at the particular relation which suggests the construction. It is not easy to see what is to suggest the producing of  $OB$  to  $D$  as in the figure.

Subjoined are a few problems of no great difficulty, which may be solved by this method.

#### PROBLEMS.

1. On a given straight line to describe a square.
2. Describe a rectangle with given sides.
3. Given two sides of a parallelogram and the included angle, construct the parallelogram.
4. Given the lengths of the two diagonals of a rhombus, construct it.
5. From a given point without a given straight line to draw a line making an angle with the line equal to a given angle.



6. Describe a square on a given straight line as diagonal.

7. Draw through a given point, between two straight lines not parallel, a straight line which shall be bisected in that point.

8. Place a line of given length between two intersecting lines so as to be parallel to another given line.

9. Trisect a right angle.

10. Divide half a right angle into six equal parts.

11. Three straight lines meet in a point, draw a straight line such that the parts of it intercepted by the three lines shall be equal to one another.

12. Trisect a given straight line.

## BOOK II.\*

### EQUALITY OF AREAS.

#### SECTION I.

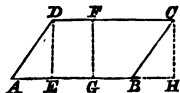
##### THEOREMS.

IN Book I. Theorems 5, 7, 15, 17, 20 we have had instances of figures whose areas are equal, and whose areas are proved to be equal, by shewing that the figures could be placed so as to coincide with one another, or are *congruent*, or identically equal. But figures of different shapes may nevertheless be equal in area, though they cannot be placed so as to coincide with one another; thus a circular field may be as large as a square one, and a triangle as large as a rectangle.

In the present section we proceed to the consideration of rectilinear figures whose areas are equal, though the figures are not of the same shape.

*Def. 1.* The *altitude* of a parallelogram with reference to a given side as base is the perpendicular distance between the base and the opposite side.

Thus in the figure the perpendiculars  $DE$ ,  $FG$ , or  $CH$ , which are equal (by I. 28) since  $DEFG$ ,  $DEHC$  are parallelograms, are each of them the altitude of the parallelogram  $ABCD$ ,  $AB$  being the base.



\* Book III. (with the exception of its last Section) is independent of Book II., and may be studied immediately after Book I.

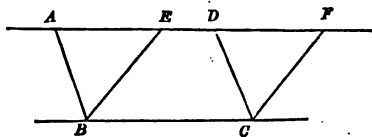
*Def. 2.* The *altitude* of a triangle with reference to a given side as base is the perpendicular distance between the base and the opposite vertex.

*Obs.* It follows from the General Axioms (*d*) and (*e*) (page 1), as an extension of the Geometrical Axiom 1 (page 11), that magnitudes which are either the sum or the difference of identically equal magnitudes are equal, although they may not be identically equal.

## THEOREM I.

*Parallelograms on the same base and between the same parallels are equal.*

*Part. En.* Let  $ABCD$ ,  $EBCF$  be parallelograms, upon the same base  $BC$ , and between the same parallels  $AF$ ,  $BC$ ;



it is required to prove that the parallelogram  $ABCD$  is equal to the parallelogram  $EBCF$ .

*Proof.* Because  $ABCD$  is a parallelogram, (Hyp.)  
 therefore  $AB = DC$ ; (1. 28.)  
 because  $AB$ ,  $BE$  are respectively parallel to  $CD$ ,  $CF$ , (Hyp.)  
 therefore the angles at  $A$  and  $E$  are respectively equal to  
 the corresponding angles at  $D$  and  $F$ ; (1. 23, Cor.)  
 therefore the triangles  $ABE$ ,  $DCF$  are equal. (1. 17.)

But if the triangle  $CDF$  is taken away from the trapezium  $ABCF$  the parallelogram  $ABCD$  remains; and if the triangle  $ABE$  is taken away from the same trapezium the parallelogram  $EBCF$  remains; therefore the parallelogram  $ABCD$  is equal to the parallelogram  $EBCF$ \*.

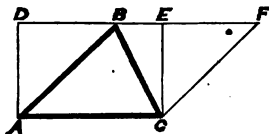
COR. 1. *The area of a parallelogram is equal to the area of a rectangle, whose base and altitude are equal to those of the parallelogram.*

COR. 2. *Parallelograms on equal bases and of equal altitude are equal†; and of parallelograms of equal altitudes, that is the greater which has the greater base; and also of parallelograms on equal bases, that is the greater which has the greater altitude.*

#### THEOREM 2.

*The area of a triangle is half the area of a rectangle whose base and altitude are equal to those of the triangle.*

Let  $ABC$  be a triangle on the base  $AC$ , and  $DACE$  the rectangle on the same base, and having the same altitude as the triangle;



then will the area of the triangle  $ABC$  be half that of the rectangle  $DACE$ .

\* Euclid, I. 35.

† Euclid, I. 36.

*Constr.* Through  $C$  draw  $CF$  parallel to  $AB$ , to meet  $DE$  produced in  $F$ .

Then  $BACF$  is a parallelogram, and therefore  $BACF$  is equal to the rectangle  $DACE$ . (II. 1.)

But the triangle  $ABC$  is half the parallelogram  $BACF$ ;  
(I. 28.)

therefore the triangle  $ABC$  is half the rectangle  $DACE$ .

COR. 1. *Triangles on the same or equal bases and of equal altitude are equal\*.*

COR. 2. *Equal triangles on the same or equal bases have equal altitudes,*

COR. 3. *If two equal triangles stand on the same base and on the same side of it, or on equal bases in the same straight line and on the same side of that straight line, the line joining their vertices is parallel to the base or to that straight line†.*

### THEOREM 3.

*The area of a trapezium is equal to the area of a rectangle whose base is half the sum of the two parallel sides, and whose altitude is the perpendicular distance between them.*

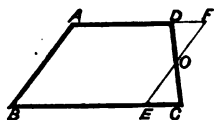
*Part. En.* Let  $ABCD$  be a trapezium having  $AD$  parallel to  $BC$ ;

then it is required to prove that its area is equal to

\* Euclid, I. 37, 38.

† Euclid, I. 39, 40.

that of the rectangle whose base is half the sum of  $AD$  and  $BC$ , and altitude the perpendicular distance between  $AD$  and  $BC$ .



*Proof.* Bisect  $DC$  in  $O$ , and through  $O$  draw a line parallel to  $AB$  to meet  $BC$  in  $E$ , and  $AD$  produced in  $F$ .

Then in the triangles  $DOF$ ,  $EOC$ ,

because	$DO = OC$ ,	(Constr.)
and the angle $DOF$	= the angle $EOC$ ,	(I. 4.)
and the angle $ODF$	= the angle $OCE$ ;	(I. 22.)

therefore the triangles are equal in all respects; (I. 7.)

and therefore the trapezium  $ABCD$  is equal to the parallelogram  $ABEF$ .

But the parallelogram  $ABEF$  is equal to the rectangle on the same base  $BE$ , and between the same parallels;

(II. 1. Cor.)

and since  $EC = DF$ ,

and  $AF = BE$ ,

therefore the base  $BE$  is half the sum of  $AD$  and  $BC$ ;

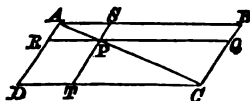
therefore the trapezium  $ABCD$  is equal to the rectangle whose base is half the sum of the parallel sides, and height the perpendicular distance between them.

*Def. 3.* The straight lines drawn through any point in a diagonal of a parallelogram parallel to the sides divide it into four parallelograms, of which the two whose diagonals are upon the given diagonal are called *parallelograms about that diagonal*, and the other two are called the *complements* of the parallelograms about the diagonal.

## THEOREM 4.

*The complements of parallelograms about the diagonal of any parallelogram are equal to one another.*

Let  $ABCD$  be a parallelogram,  $P$  any point on the diagonal  $AC$ , and let  $RPQ$ ,  $SPT$  be drawn parallel to the sides;



it is required to prove that the complement  $PB$  = the complement  $PD$ .

*Proof.* For the triangle  $ABC$  = the triangle  $ADC$  (I. 28); and the triangles  $ASP$ ,  $PQC$  = the triangles  $ARP$ ,  $PTC$ ; therefore the remainders are equal, that is,  $PB = PD$ \*.

*Def. 4.* All rectangles being identically equal which have two adjoining sides equal to two given straight lines, any such rectangle is spoken of as *the rectangle contained by those lines*.

In like manner, any square whose side is equal to a given straight line is spoken of as *the square on that line*.

*Def. 5.* A point in a straight line is said to divide it *internally*, or, simply, to divide it; and, by analogy, a point


\* Euclid, I. 43.

in the line produced is said to divide it *externally*; and, in either case, the distances of the point from the extremities of the line are called its *segments*.

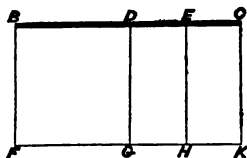
*Obs.* A straight line is equal to the sum or difference of its segments according as it is divided internally or externally.

## THEOREM 5.

*The rectangle contained by two given lines is equal to the sum of the rectangles contained by one of them and the several parts into which the other is divided.*

*Part. En.* Let  $A$  and  $BC$  be the two given lines, of which  $BC$  is divided into any number of parts,  $BD, DE, EC$ ; 

it is required to prove that the rectangle contained by  $A$  and  $BC$  is equal to the sum of the rectangles contained by  $A$  and  $BD$ ,  $A$  and  $DE$ ,  $A$  and  $EC$ .



*Proof.* From  $B$  draw a line  $BF$  at right angles to  $BC$ , and equal to  $A$ ; through  $F$  draw a line parallel to  $BC$ ; and through  $D, E, C$  draw  $DG, EH, CK$  parallel to  $BF$ .

Then the figure  $BK$  is equal to the figures  $BG, DH, EK$ :

but  $BK$  is the rectangle contained by  $A$  and  $BC$ ;

and  $BG, DH, EK$  are respectively the rectangles contained by  $A$  and  $BD$ ,  $A$  and  $DE$ ,  $A$  and  $EC$ ;



therefore the rectangle contained by  $A$  and  $BC$  is equal to the rectangles contained by  $A$  and  $BD$ ,  $A$  and  $DE$ ,  $A$  and  $EC^*$ .

COR. 1. *If a straight line is divided into two parts, the rectangle contained by the whole line and one of the parts is equal to the sum of the square on that part and the rectangle contained by the two parts†.*

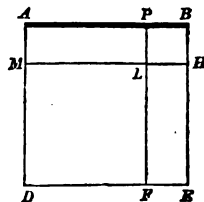
COR. 2. *If a straight line is divided into two parts the square on the whole line is equal to the sum of the rectangles contained by the whole line and each of the parts‡.*

#### THEOREM 6.

*The square on the sum of two lines is greater than the sum of the squares on those lines by twice the rectangle contained by them§.*

*Part. En.* Let  $AB$  be the sum of  $AP$ ,  $PB$ ;

it is required to prove that the square on  $AB$  is equal to the squares on  $AP$ ,  $PB$  together with twice the rectangle contained by  $AP$ ,  $PB$ .



*Proof.* Describe a square  $ADEB$  on  $AB$ .

Through  $P$  draw  $PLF$  parallel to  $AD$ , meeting  $DE$  in  $F$ ; cut off  $PL = PB$  leaving  $LF = AP$ . Through  $L$  draw  $HLM$  parallel to  $AB$ , to meet  $DA$  and  $EB$  in  $M$ ,  $H$ .

\* Euclid, II. 1. † Euclid, II. 3. ‡ Euclid, II. 2. § Euclid, II. 4.

Then the figures  $AL$ ,  $PH$ ,  $LE$ ,  $MF$  are rectangles by construction;

and  $PH$ ,  $MF$  are the squares on  $PB$ ,  $AP$  respectively; and  $AL$ ,  $LE$  are each of them the rectangle contained by  $AP$ ,  $PB$ .

Hence, since  $ADEB$  is made up of these four figures, it follows that the square on  $AB$  is greater than the squares on  $AP$ ,  $PB$  by twice the rectangle contained by  $AP$ ,  $PB$ .

## THEOREM 7.

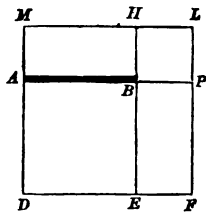
*The square on the difference of two lines is less than the sum of the squares on those lines by twice the rectangle contained by them\*.*

*Part. En.* Let  $AB$  be the difference of  $AP$ ,  $BP$ ; it is required to prove that the square on  $AB$  is less than the squares on  $AP$ ,  $PB$  by twice the rectangle  $AP$ ,  $PB$ .

*Proof.* Describe a square  $ADEB$  on  $AB$ .

Through  $P$  draw  $LPF$  parallel to  $AD$ , meeting  $DE$  produced in  $F$ : cut off  $PL=PB$ , making  $LF=AP$ .

Through  $L$  draw  $MHL$  parallel to  $AB$ , to meet  $DA$  and  $EB$  produced in  $M$ ,  $H$ .



Then  $MF$  is the square on  $AP$ ; and  $HP$  the square on  $BP$ ; and  $MP$  or  $HF$ , the rectangle contained by  $AP$  and  $BP$ .

\* Euclid, II. 7.

And  $AE$  is less than  $MF + HP$  by  $MP + HF$ ; that is, the square on  $AB$  is equal to the squares of  $AP, PB$  diminished by twice the rectangle contained by  $AP, PB$ .

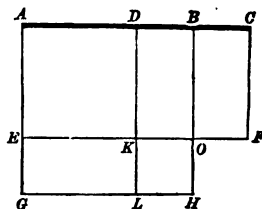
## THEOREM 8.

*The difference of the squares on two lines is equal to the rectangle contained by the sum and difference of the lines\*.*

*Part. En.* Let  $AB$  and  $BC$  be the two straight lines, of which  $AB$  is the greater; and let them be placed in one straight line; cut off  $BD$  equal to  $BC$ ; so that  $AC$  is their sum, and  $AD$  is their difference;

Then will the difference of the squares of  $AB$  and  $BC$  be equal to the rectangle contained by  $AC$  and  $AD$ .

*Proof.* On  $AB$  describe a square  $AGHB$ . Through  $D$ ,  $C$  draw  $DL, CF$  parallel to  $AG$  or  $BH$ ; cut off  $HO = LH$  or  $DB$ ; and through  $O$  draw  $EKOF$  parallel to  $AC$ .



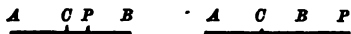
Then  $KH$  is the square on  $DB$  or  $BC$ ; and therefore the difference of the squares of  $AB$  and  $BC$  is the figure made up of  $EL$  and  $AO$ .

But  $EL$  is equal to  $BF$  by construction; therefore the figure made of  $EL$  and  $AO$  is equal to  $AF$ ; which is the rectangle contained by  $AE$  or  $AD$  and  $AC$ ;

\* Euclid, II. 5, Cor.

therefore the difference of the squares of  $AB$  and  $BC$  is equal to the rectangle contained by  $AC$  and  $AD$ .

*COR.* If a straight line is bisected and divided in any point, the rectangle contained by the segments is equal to the difference of the squares on half the line and the line between the points of section.



*Proof.* For let  $AB$  be bisected in  $C$ , and divided internally or externally in  $P$ .

Then  $AP$  is the sum of  $AC$  and  $CP$ , and  $PB$  is their difference, since  $BC = AC$ .

Therefore the rectangle contained by  $AP$ ,  $PB$  is the rectangle contained by the sum and difference of  $AC$  and  $CP$ , and therefore is equal to the difference of the squares of  $AC$  and  $CP$ .

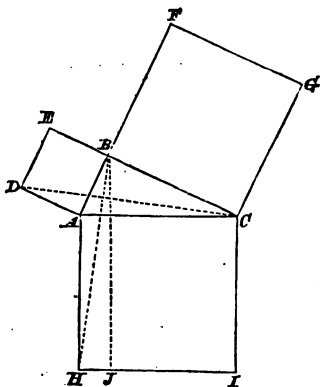
*Remark.* The student will begin here to suspect, what he will afterwards find to be true, that there is an intimate relation between geometry and algebra. Algebraical or analytical geometry as it is called, investigates this relation and applies it to the establishment of theorems in geometry, and will occupy him at a later stage of his mathematical studies. We shall at present use the expression  $AB^2$ , which is read ' $AB$  squared,' only as an abbreviation for "the square on  $AB$ ," and  $AB \times AC$  or  $AB.AC$ , as an abbreviation for "the rectangle contained by  $AB$  and  $AC$ ."

#### THEOREM 9.

*In any right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the sides which contain the right angle.*

*Part. En.* Let  $ABC$  be a triangle right-angled at  $B$ ; it is required to prove that  $AC^2$  is equal to  $AB^2 + BC^2$ .

*Proof.* On  $AB$ ,  $BC$ ,  $CA$  describe the squares  $ADEB$ ,  $BFGC$ ,  $CIHA$  respectively. Join  $CD$ ,  $BH$ ; and draw  $BJ$  parallel to  $AH$ .



Since the angles  $ABC$ ,  $ABE$ ,  $BCF$  are right angles; it follows that  $CBE$ ,  $ABF$  are straight lines; (I. 3.) therefore the triangle  $DAC$  is on the same base  $DA$ , and between the same parallels  $DA$ ,  $EC$  with the square  $DABE$ ; therefore the triangle  $DAC$  is half the square  $DABE$ ;  
(II. 3, Cor. 2.)

and similarly the triangle  $BAH$  is half the rectangle  $AJ$ .

But because the angle  $DAB$  = the angle  $HAC$ , each being a right angle; add to each the angle  $BAC$ ;

therefore the whole angle  $DAC$  is equal to the whole angle  $BAH$ ;

and the two sides  $DA$ ,  $AC$  are respectively equal to the two sides  $BA$ ,  $AH$ ;  
(Constr.)

therefore the triangle  $DAC$  is equal to  $BAH$ ;  
(I. 5.)

and therefore the square  $DABE$  = the rectangle  $AJ$ .

Similarly it may be shewn that the square  $BFGC$  = the rectangle  $CJ$ , and therefore, since  $AJ$  and  $CJ$  make up

the whole square  $AHIC$ , the square  $AHIC$  is equal to the sum of the squares  $ABDE$  and  $BCGF$ , that is,

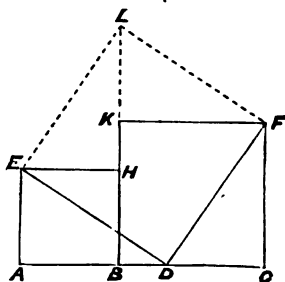
$$AC^2 = AB^2 + BC^2.*$$

This important proposition may be proved as follows.

Place two squares  $EABH$ ,  $KBCF$ , as in the figure, with their sides  $AB$ ,  $BC$  continuous and in the same straight line.

From  $CB$  cut off  $CD$  equal to  $AB$ . Join  $DE$ ,  $DF$ .

Produce  $BK$  to  $L$ , making  $KL = AB$ . Join  $LE$ ,  $LF$ .



Then it will be easy to prove that the triangles  $EAD$ ,  $DCF$ ,  $EHL$ ,  $LKF$  are all equal, being right-angled, and having the sides containing the right angle equal; therefore the figure  $LEDF$  is equal to the sum of the two given squares, and all its sides are equal.

And since  $EDA$  is complementary to  $AED$  or  $FDC$ , therefore the angle  $EDF$  is a right angle.

Therefore  $LEDF$  is a square, and is the square on  $ED$ .

Therefore the square on the hypotenuse  $ED$  is equal to the sum of the squares on the sides  $EA$ ,  $AD$ .

\* Euclid, I. 47.

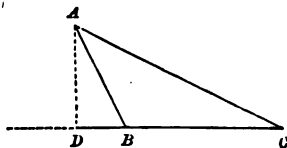
COR. 1. *It follows that in a triangle ABC right-angled at B,*

$$AB^2 = AC^2 - BC^2 \text{ and } BC^2 = AC^2 - AB^2.$$

The next two theorems shew the modifications which the theorem undergoes when the triangle is not right-angled.

### THEOREM 10.

*In an obtuse-angled triangle the square on the side subtending the obtuse angle is greater than the squares on the sides containing that angle by twice the rectangle contained by either of these sides and the projection on it of the other side\*.*



*Part. En.* Let  $ABC$  be the triangle,  $ABC$  being the obtuse angle,  $BD$  the projection of  $AB$  on  $BC$ ,  $BC$  being produced backward.

Then will  $AC^2 = AB^2 + BC^2 + 2CB \cdot BD$ ,  
 for  $AC^2 = AD^2 + DC^2$ , by (II. 9.)  
 but  $AD^2 = AB^2 - BD^2$ ,  
 and  $DC^2 = CB^2 + BD^2 + 2CB \cdot BD$ , (by II. 6.)  
 therefore  $AC^2 = AB^2 + BC^2 + 2CB \cdot BD$ .

\* Euclid, II. 12.

## THEOREM 11.

*In any triangle the square on the side opposite an acute angle is less than the squares on the other two sides by twice the rectangle contained by either side and the projection on it of the other side\*.*

*Part. En.* Let  $ABC$  be a triangle,  $B$  an acute angle,  $BD$  the projection of  $AB$  on  $BC$ , then will

$$AC^2 = AB^2 + BC^2 - 2CB \times BD.$$



*Proof.* For  $AC^2 = AD^2 + DC^2$ , (by II. 9);  
 but  $AD^2 = AB^2 - BD^2$ , by the same Theorem,  
 and  $DC^2 = BC^2 + BD^2 - 2CB \times BD$ , (by II. 7);  
 therefore  $AC^2 = AB^2 + BC^2 - 2CB \times BD$ .

*COR.* Conversely, the angle opposite a side of a triangle is an acute angle, a right angle, or an obtuse angle, according as the square on that side is less than, equal to, or greater than the sum of the squares on the other two sides.

## THEOREM 12.

*The sum of the squares on two sides of a triangle is double the sum of the squares on half the base and on the line joining the vertex to the middle point of the base.*

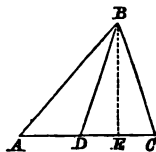
\* Euclid, II. 13.



*Part. En.* Let  $AC$ , a side of the triangle  $ABC$ , be bisected in  $D$ ; then will

$$AB^2 + BC^2 = 2AD^2 + 2BD^2.$$

*Proof.* For let  $DE$  be the projection of  $BD$  on  $AC$ .



Then  $AB^2 = AD^2 + DB^2 + 2AD \cdot DE$  (by II. 10),  
and  $BC^2 = CD^2 + DB^2 - 2CD \cdot ED$  (by II. 11),  
therefore remembering that  $AD = DC$ , we obtain by addition that

$$AB^2 + BC^2 = 2AD^2 + 2DB^2.$$

This theorem in a more general form is known as Apollonius's Theorem.

#### THEOREM 13.

*If a straight line is divided internally or externally at any point, the sum of the squares on the segments is double the sum of the squares on half the line and on the line between the point of division and the middle point of the line\*.*

Let  $AB$  be bisected in  $C$ , and divided internally or externally in  $D$ .



Then the squares on  $AD$ ,  $DB$  will be double of the squares on  $AC$ ,  $CD$ .

*Proof.* For  $AD^2 = AC^2 + CD^2 + 2AC \times CD$  by II. 6;

and  $DB^2 = CB^2 + CD^2 - 2BC \times CD$  by II. 7;

therefore, adding, and remembering that  $AC = BC$ , and that therefore the rectangle  $AC \times CD =$  the rectangle  $BC \times CD$ , we get that  $AD^2 + DB^2 = 2AC^2 + 2CD^2$ .

\* Eucl. II. 9, 10.

## EXERCISES.

1. Bisect a triangle by a line passing through one of its angular points.

2. Any line drawn through the intersection of the diagonals of a parallelogram to meet the sides bisects the figure.

3. Find the locus of the vertices of triangles of equal area upon the same base.

4. If the sides of a triangle are 3, 4, 5 inches respectively, the triangle is right-angled.

5. Of all triangles having the same vertical angle, and whose bases pass through a given point, the least is that whose base is bisected in that point.

6. The diagonals of a parallelogram divide it into four equivalent triangles.

7. If from any point in the diagonal of a parallelogram straight lines be drawn to the angles, then the parallelogram will be divided into two pairs of equivalent triangles.

8.  $ABCD$  is a parallelogram, and  $E$  any point in the diagonal  $AC$  produced. Shew that the triangles  $EBC$ ,  $EDC$  will be equivalent.

9.  $ABCD$  is a parallelogram, and  $O$  any point within it, shew that the triangles  $OAB$ ,  $OCD$  are together equivalent to half the parallelogram.

10. On the same supposition if lines are drawn through  $O$  parallel to the sides of the parallelogram, then the difference of the parallelograms  $DO$ ,  $BO$  is double of the triangle  $OAC$ .

11. The diagonals of a parallelogram  $ABCD$  intersect in  $O$ , and  $P$  is a point within the triangle  $OAB$ . Prove that the difference of the triangles  $APB$ ,  $CPD$ , is equivalent to the sum of the triangles  $APC$ ,  $BPD$ .

12. If the points of bisection of the sides of a triangle be joined, the triangle so formed shall be one-fourth of the given triangle.

13. Shew that the sum of the squares on the lines joining the angular points of a square to any point within it is double of the sum of the squares on the perpendiculars from that point on the sides.

14. If the sides of a quadrilateral figure be bisected, and the points of bisection joined, prove that the figure so formed will be a parallelogram equal in area to half the given quadrilateral.

15. Bisect a parallelogram by a line passing through any given point.

## SECTION II.

### PROBLEMS.

#### *On the Quadrature of a Rectilineal Area.*

There is one problem which from its historical interest, and from the valuable illustrations it affords of the methods and limitations of Geometry, should find a place there. This problem is called *the quadrature of a rectilineal area*, which means the finding a square whose area is equivalent to that of any given figure which is bounded by straight lines. It gave a means of comparing any two dissimilarly

shaped rectilinear figures, such as irregularly shaped fields whose boundaries were straight. In the present condition of mathematics it is not necessary, as the student will hereafter learn, but it will always be instructive.

The problem is approached by the following stages:

(1) To construct a parallelogram, with sides inclined at a given angle, equal to a given triangle.

(2) To construct *on a given straight line* a parallelogram, with sides inclined at a given angle, equal to a given triangle.

(3) To construct a parallelogram, with sides inclined at a given angle, equal to a *given rectilinear figure*.

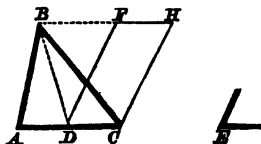
(4) To construct a *square* equal to a given rectilinear figure.

#### PROBLEM I.

*To construct a parallelogram equal to a given triangle and having one of its angles equal to a given angle.*

Let  $ABC$  be the given triangle,  $E$  the given angle.

*Construction.* Bisect  $AC$  in  $D$ , make the angle  $CDF = E$ , and through  $B$  draw  $BFH$  parallel to  $AC$ , and draw  $CH$  parallel to  $DF$ .



$FDCH$  will be the parallelogram required.

*Proof.* If  $BD$  be joined, it will be clear that the triangle  $BAC$  and the parallelogram  $FHCD$  are each of them double of the triangle  $BDC$  (II. 2, Cor. 1), and therefore the parallelogram  $FHCD =$  the triangle  $BAC$ , and it has an angle  $= E$ , which was required\*.

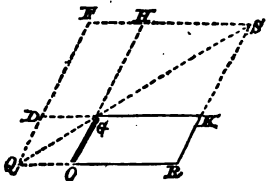
\* Euclid, I. 42.

## PROBLEM II.

*To construct a parallelogram on a given base equal to a given triangle and having one of its angles equal to a given angle.*

Let  $BAC$  be the given triangle,  $E$  the given angle as before, and let it be required to construct on the line  $GO$  a parallelogram equal to  $BAC$ , and having an angle  $E$ .

*Construction.* Construct the parallelogram  $FDGH$  as before, and place it so that one of its sides  $GH$  may be in the same straight line with  $GO$ .



Produce  $FD$ , and draw  $OQ$  parallel to  $GD$  to meet  $FD$  in  $Q$ .

Join  $QG$ , and produce it to meet  $FH$  produced in  $S$ .

Draw  $SKR$  parallel to  $FQ$ , meeting  $DG$  produced in  $K$ , and  $QO$  produced in  $R$ .

Then  $GORK$  is the parallelogram required.

*Proof.* For the parallelogram  $FG$  = the parallelogram  $GR$ , being complements (II. 4); and  $FG$  = the given triangle  $ABC$ .

Therefore  $GR$  = the triangle  $ABC$ , and it has an angle  $E$ , since it is equiangular with the parallelogram  $FDGH$ \*.

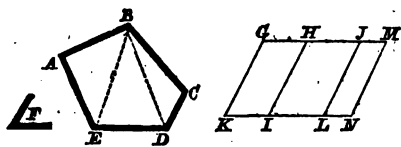
## PROBLEM III.

*To construct a parallelogram equal to a given rectilineal figure and having one of its angles equal to a given angle.*

\* Euclid, I. 44.

Let  $ABCDE$  be the given rectilineal figure;  $F$  the given angle. Divide  $ABCDE$  into triangles by joining  $BE$ ,  $BD$ .

*Construction.* Construct as before a parallelogram  $GHIK = BAE$ , and having an angle at  $K = F$ .



Construct on  $HI$  a parallelogram  $HJLI = BED$ , and having the angle  $HIL = F$ .

And construct on  $JL$  a parallelogram  $JMNL = BCD$  and having the angle  $JLN = F$ .

$GKNM$  will then be the parallelogram required.

*Proof.* For since the angle  $HIL =$  the angle  $K$ , it is therefore supplementary to  $HIK$ ; and therefore (by 1. 3)  $KIL$  is a straight line.

Similarly  $GM$  and  $KN$  are straight lines; and  $MN$  is parallel to  $GK$ . (I. 24.)

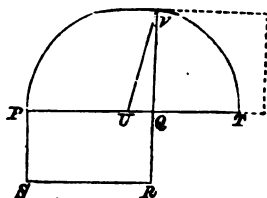
Therefore  $GKNM$  is a parallelogram, having the given angle, and it is by construction equal to the given rectilineal figure\*.

#### PROBLEM IV.

*To construct a square equal to a given rectilineal figure.*

\* Euclid, I. 45.

*Construction.* By the previous construction make a rectangle equal to  $ABCDE$ , and let  $PQRS$  be the rectangle so made.



Then if  $PQ = QR$  the rectangle is a square; but if not, produce  $PQ$  to  $T$ , making  $QT = QR$ ; on  $PQT$  as diameter describe a semicircle,  $U$  being the centre, and produce  $RQ$  to meet the circumference in  $V$ .

If a square be described on  $VQ$ , this square will be equal to  $ABCDE$ .

*Proof.* For since  $PQ$  is the sum of  $PU$  and  $UQ$ , and  $QT$  is the difference of  $PU$  (or  $UT$ ) and  $UQ$ , it follows (from II. 8) that the rectangle  $PQ \times QT = PU^2 - UQ^2$ ; but  $PU^2 = UV^2$ , and therefore  $PU^2 - UQ^2 = UV^2 - UQ^2$ , that is,  $VQ^2$ , by II. 9, Cor.

But the rectangle  $PQ \times QT$  is the rectangle  $PQRS$ , which was made equal to  $ABCDE$ .

Therefore  $VQ^2 = ABCDE$ , and the square described on  $VQ$  is the square required\*.

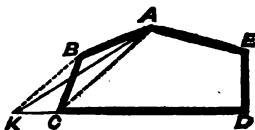
*Remark.* If the given figure is not rectilinear, it cannot be divided into triangles: hence it is impossible by this method to construct a square equal to a given curvilinear area. Nor can any method depending on the use of the ruler and compasses only (see p. 60), construct a square equal to some curvilinear areas, such as the circle. This is the problem of squaring the circle, the solution of which cannot be effected without the use of other instruments.

\* Euclid, II. 14.

## PROBLEM V.

*To construct a rectilinear figure equal to a given rectilinear figure and having the number of its sides one less than that of the given figure; and thence to construct a triangle equal to a given rectilinear figure.*

Let  $ABCDE$  be the given rectilinear figure. Join  $AC$ , and through  $B$  draw  $BK$  parallel to  $AC$  to meet  $DC$  produced in  $K$ . Join  $AK$ .



Then since the triangle  $ABC$  is equal to the triangle  $AKC$ , being on the same base  $AC$  and between the same parallels; add to each  $ACDE$ ; therefore the figure  $ABCDE$  is equal to the figure  $AKDE$ , which has the number of its sides diminished by one.

Since this process can be repeated any number of times it is evident that any polygon can be reduced in this manner to an equivalent triangle.

## PROBLEM VI.

*To divide a straight line, either internally or externally, into two segments such that the rectangle contained by the given line and one of the segments may be equal to the square on the other segment.*

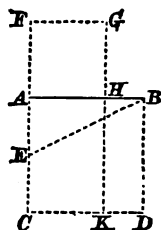


Let  $AB$  be the given line.

First, to divide it internally.

*Construction.* Draw a square  $ACDB$  on  $AB$ ; bisect  $AC$  in  $E$ . Join  $BE$ ; produce  $EA$  to  $F$ , making  $EF = EB$ ; on  $AF$  describe a square  $AFGH$ .

$AH$  and  $HB$  are the parts required, so that the rectangle  $AB \times BH = AH^2$ .



*Proof.* Produce  $GH$  to meet  $CD$  in  $K$ .

Then since  $CA$  is bisected in  $E$ , and divided externally in  $F$ ,

$$\text{therefore } CF \times FA = EF^2 - EA^2 \text{ (II. 8. Cor.)};$$

$$\text{but } EF^2 = EB^2, \text{ and therefore } EF^2 - EA^2 = AB^2 \quad (\text{II. 9. Cor.}),$$

$$\text{therefore } CF \times FA = AB^2;$$

that is, the figure  $FK$  = the figure  $AD$ ; take from each  $AK$ , and therefore  $FH = HD$ .

But  $HD$  is the rectangle  $AB \times BH$ ; and  $FH$  is the square on  $AH$ ;

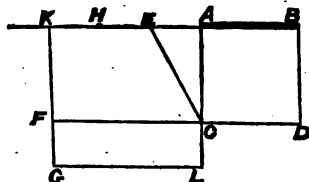
$$\text{therefore } AB \times BH = AH^2.*$$

*COR.* The line  $FA$  is divided externally in  $C$ , so that

$$FC \cdot FA = CA^2.$$

\* Euclid, II. 11.

Secondly, to divide it externally.



*Construction.* Produce  $BA$ , and take  $AH$  equal to  $AB$ . Bisect  $AH$  in  $E$ . On  $AB$  describe a square  $ACDB$ . Join  $EC$ ; from  $EH$  produced, cut off  $EK = EC$ .

Then will  $AB \cdot BK = AK^2$ .

*Proof.* On  $KA$  describe a square  $KGLA$ , and produce  $DC$  to meet  $KG$  in  $F$ .

Then, since  $AH$  is bisected in  $E$  and produced to  $K$ ,

$$AK \times KH = EK^2 - EA^2; \quad (\text{II. 8.})$$

but  $EK^2 = EC^2$ , and therefore  $EK^2 - EA^2 = AC^2$ ,

therefore  $AK \times KH = AC^2$ ;

that is, the figure  $FL =$  the figure  $AD$ .

Add to each  $KC$ ,

then the figure  $KL =$  the figure  $KD$ ;

that is, the square on  $AK$  is equal to the rectangle  $AB \times BK$ .

Q. E. D.

### EXERCISES.

1. Construct a square double of a given square.
2. Construct a square equal to two, or three, or any number of given squares.

3. Divide a straight line into two parts, such that the square of one of the parts may be half the square on the whole line.

4. Given the base, area, and one of the angles at the base, construct the triangle.

5. Find the locus of a point which moves so that the sum of the squares of its distances from two given points is constant.

We subjoin a few problems and theorems as miscellaneous exercises in the Geometry of angles, lines, triangles, parallelograms, and the equality of areas.

#### MISCELLANEOUS THEOREMS AND PROBLEMS.

1. Prove that the acute angle between the bisectors of the angles at the base of an isosceles triangle is equal to one of the angles at the base of the triangle.

2. Find a point equally distant from three given straight lines.

3. If the diagonals of a quadrilateral bisect one another and are equal to one another, the figure will be a rectangle.

4. If the diagonals of a quadrilateral bisect one another at right angles and are also equal, the figure will be a square.

5. If  $ABC$  is a triangle,  $AB$  being greater than  $AC$ , and a point  $D$  in  $AB$  be taken such that  $AD = AC$ ; prove that the angle  $BCD$  is equal to half the difference of the angles  $ABC, ACB$ .

6. If  $ABCD$  is a parallelogram, and  $AE = CF$  are cut off from the diagonal  $AC$ , then  $BEDF$  will be a parallelogram.

7. If  $AA' = CC'$  be cut off from the diagonal  $AC$ , and  $BB' = DD'$  from the diagonal  $BD$  of a parallelogram, then will  $A'B'C'D'$  be also a parallelogram.

8. If  $AA' = BB' = CC' = DD'$  be cut off from the sides of the parallelogram  $ABCD$  taken in order, then will  $A'B'C'D'$  be also a parallelogram.

9.  $ABC$  is a triangle, and through  $D$ , the middle point of  $AB$ ,  $DE, DF$  are drawn parallel to the sides  $BC, AC$ , to meet them in  $E, F$ . Shew that  $EF$  is parallel to  $AB$ .

10. Through a given point to draw a line such that the part of it intercepted between two parallel lines shall have a given length.

11. To describe a rhombus equal to a given parallelogram, having its side equal to the longer side of the parallelogram.

12. Shew that the diagonal of a rectangle is longer than any other line whose extremities are on the sides of the rectangle.

13. From the extremities of the base of an isosceles triangle straight lines are drawn perpendicular to the opposite sides; the angles made by them with the base are equal to half the vertical angle.

14.  $D$  is the middle point of the side  $AC$  of a triangle  $ACB$ , and any parallel lines  $BE$ ,  $DF$  are drawn to meet  $AC$ ,  $AB$  (or  $BC$ ) in  $E$  and  $F$ , shew that  $EF$  divides the triangle into two equal areas.

15. If every pair of alternate sides of a convex figure of five sides be produced to meet, so as to form a five-rayed star, prove that the angles so formed will be together equal to two right angles.

Extend this to the case of a polygon of  $n$  sides.

16. Of all triangles having the same base and area, that which is isosceles has the least perimeter.

17. The area of a rhombus is equal to half the rectangle constructed on the two diameters of the rhombus.

18. If two opposite sides of a quadrilateral are parallel, and their points of bisection joined, the quadrilateral will be bisected.

19. If two opposite sides of a parallelogram be bisected, and lines be drawn from these two points of bisection to the opposite angles, these lines will be parallel, two and two, and will trisect both diagonals.

20. The sum of the squares described on the sides of a rhombus is equal to the squares described on its diameters.

21. From the sides of the triangle  $ABC$ ,  $AA'$ ,  $BB'$ ,  $CC'$ , are cut off each equal to two-thirds of the side from which it is cut. Shew that the triangle  $A'B'C'$  is one-third of the triangle  $ABC$ .

22.  $BCD...$  are points on the circumference of a circle,  $A$  any point not the centre of the circle. Shew that of the lines  $AB$ ,  $AC$ ,  $AD...$  not more than two can be equal.

23. Find the locus of a point, such that the sum of the squares on its distances from two given points is equal to the square on the distance between the two points.

24. If  $m$  and  $n$  are any numbers, and lines be taken whose lengths are  $m^2 + n^2$ ,  $m^2 - n^2$  and  $2mn$  units respectively, shew that these lines will form a right-angled triangle. Give examples of these triangles.

25. Through two given points on opposite sides of a straight line draw two straight lines to meet in that line, so that the angle which they form shall be bisected by that line.

26. Through a given point draw a line such that the perpendiculars on it from two given points may be equal.

27. Find points  $D, E$  in the equal sides  $AB, AC$  of an isosceles triangle  $ABC$ , such that  $BD = DE = EC$ .

28. If one angle of a triangle is equal to the sum of the other two, the greatest side is double of the distance of its middle point from the opposite angle.

29. Find the locus of a point, given the sum or difference of its distances from two fixed lines.

30. Given two points and a straight line of indefinite length, construct an equilateral triangle so that two of its sides shall pass through the given points, and the third shall be in the given straight line.

31. Construct an isosceles triangle having the angle at the vertex double of the angles at the base.

32.  $ABC$  is a triangle,  $AB$  greater than  $BC$ ;  $BD$  bisects the base  $AC$ , and  $BE$  the angle  $ABC$ . Prove (1) that  $ADB$  is an obtuse angle; (2) that  $ABD$  is less than  $DBC$ ; and (3) that  $BE$  is less than  $BD$ .

33. Bisect a triangle by a line passing through a point in one of its sides.

34. If two sides of a triangle be given, its area will be greatest when they contain a right angle.

35. Construct a triangle equal to a given quadrilateral figure.

36. Bisect a given quadrilateral figure by a line drawn from one of its angular points.

37. Bisect a given five-sided figure by a line drawn from one of its angular points.

38. If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.

39. Produce a given straight line to such a distance that the square on the produced part may be double of the square on the given line.

40. Produce a given straight line to such a distance that the square on the whole line may be double of the square on the given line.

41. Given two sides and a median, construct the triangle.

42. Divide a straight line into two parts such that the square on one part may be four times the square on the other.

43. From  $B$ , one of the angles of a triangle  $ABC$ , a perpendicular  $BD$  is let fall on  $AC$ . Shew that the difference of the squares on  $AB$ ,  $BC$  is equal to the difference of the squares on  $AD$ ,  $DC$ .

44.  $AC$  one of the sides of a triangle  $ABC$  is bisected in  $D$ : and  $BD$  joined. Shew that the squares on  $AB$  and  $BC$  together are equal to twice the square on  $BD$ , and twice the square on  $AD$ .

45. Produce a given line  $AB$  to  $P$  so that  $AP \cdot BP = AB^2$ .

46.  $ABCD$  is the diameter of two concentric circles,  $P, Q$  any points on the outer and inner circles respectively. Prove that  $BP^2 + CP^2 = AQ^2 + DQ^2$ .

47. Given a polygon of  $n$  sides to construct an equal polygon of  $(n-1)$  sides. Hence construct a rectangle equal to any given rectilineal figure.

48. Prove that the squares on the diagonals of any parallelogram are together equal to the squares on its sides.

49.  $O$  is the point of intersection of the diagonals of a square  $ABCD$ , and  $P$  any other point whatever. Prove that  $AP^2 + BP^2 + CP^2 + DP^2 = 4OA^2 + 4OP^2$ .

50. Given the base, difference of sides, and difference of the angles at the base, construct the triangle.

51. If from one of the acute angles of a right-angled triangle a line be drawn to the opposite side, the squares on that side and the line so drawn are together equal to the squares on the segment adjacent to the right angle and on the hypotenuse.

52. Find the locus of the middle point of a line drawn from a given point to meet a given line.

53. If from the right angle  $C$  of a right-angled triangle  $ABC$  straight lines be drawn to the opposite angles of the square on  $AB$ , the difference of the squares on these two lines will equal the difference of the squares on  $AC$  and  $BC$ .



54.  $AB$  is divided into two unequal parts in  $C$  and equal parts in  $D$ ; shew that the squares on  $AC$  and  $BC$  are greater than twice the rectangle  $AC \times CB$  by four times the square on  $CD$ .

55. In any right-angled triangle the square on one of the sides containing the right angle is equal to the rectangle contained by the sum and difference of the other two sides.

56. In any isosceles triangle  $ABC$ , if  $AD$  is drawn from  $A$  the vertex to any point  $D$  in the base, shew that

$$AB^2 = AD^2 + BD \cdot DC.$$

57. Prove that four times the sum of the squares on the medians of a triangle is equal to three times the sum of the squares on the sides of the triangle.

A medium of a triangle is the line drawn from an angle to the point of bisection of the opposite side.

58. The square on the base on an isosceles triangle is double the rectangle contained by either side, and the projection on it of the base.

59. The squares on the diagonals of a quadrilateral are double of the squares on the sides of the parallelogram formed by joining the middle points of its sides.

60. Hence shew that they are also double of the squares on the lines which join the points of bisection of the opposite sides of the quadrilateral.

61. The squares on the diagonals of a quadrilateral are together less than the squares on the four sides by four times the square on the line joining the points of bisection of the diagonals.

62. In any quadrilateral figure the lines which join the middle points of opposite sides intersect in the line which joins the middle point of the diagonals, and bisect one another at that point.

63. The locus of a point which moves so that the sum of the squares of its distances from three given points is constant is a circle.

## BOOK III.

### THE CIRCLE.

#### SECTION I.

##### ELEMENTARY PROPERTIES.

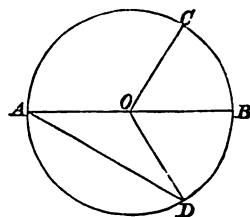
A *circle* is a plane figure contained by one line, which is called the circumference, and is such that all the lines drawn from a certain point within the figure to the circumference are equal to one another. This point is called the centre of the circle.

A straight line drawn to the circumference from the centre is called a *radius* of the circle.

A straight line drawn through the centre and terminated both ways by the circumference is called a *diameter* of the circle.

*Def. 1.* An *arc* is a part of a circumference.

*Def. 2.* A *chord* of a circle is the straight line joining any two points on the circumference. When the arcs into which the chord divides the circumference are unequal, they are called the *major* and *minor* arcs respectively. Such arcs are said to be *conjugate* to one another.



*Def. 3.* A *segment* of a circle is the figure contained by a chord and either of the arcs into which the chord divides the circumference. The segments are called *major* and *minor* segments according as the arcs that bound them are major or minor arcs.

*Def. 4.* The *conjugate* angles formed at the centre of a circle by two radii are said to *stand upon* the conjugate arcs opposite them intercepted by the radii, the major angle upon the major arc, and the minor angle upon the minor arc.

*Def. 5.* A *sector* is the figure contained by an arc and the radii drawn to its extremities. The *angle of the sector* is the angle at the centre which stands upon the arc of the sector.

*Def. 6.* Circles that have a common centre are said to be *concentric*.

The following properties of the circle are immediate consequences of Book I. Def. 8:

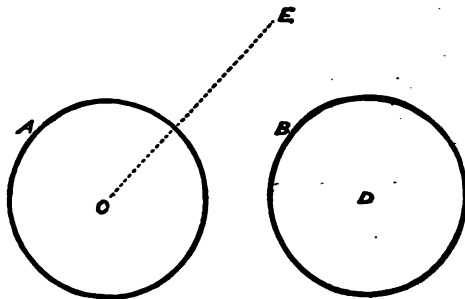
- (a) A circle has only one centre.
- (b) A point is within, on, or without the circumference of a circle, according as its distance from the centre is less than, equal to, or greater than the radius.
- (c) The distance of a point from the centre of a circle is less than, equal to, or greater than the radius, according as the point is within, on, or without the circumference.

## THEOREM I.

*Circles of equal radii are identically equal.*

*Part. En.* Let  $A$  and  $B$  be circles of equal radii; it is required to prove that they are identically equal.

*Proof.* Let their centres be  $C$  and  $D$ . Place the circle  $B$  upon the circle  $A$  so that the point  $D$  falls upon the point  $C$ , and take any point  $E$  outside both circles and



join  $CE$ . Then since all radii of the same circle are equal, and the circles are of equal radii; therefore the distances from  $C$  along  $CE$  to the circumferences of the two circles are the same; therefore the circumferences cut the line  $CE$  in the same point.

Similarly they cut every line through  $C$  in the same point, and therefore coincide altogether, and the two circles are identically equal.

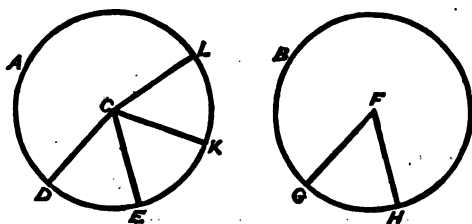
*COR.* Two (different) circles whose circumferences meet one another cannot be concentric\*.

\* Euclid, III. 5.

## THEOREM 2.

*In the same circle, or in equal circles, equal angles at the centre stand on equal arcs; and of two unequal angles at the centre the greater angle stands on the greater arc\*.*

*Part. En.* Let  $A$  and  $B$  be two equal circles, and  $DCE$ ,  $GFH$  two angles at their centres  $C$  and  $F$ , standing upon the arcs  $DE$ ,  $GH$  respectively; it is required to



prove that if the angle  $DCE$  be equal to the angle  $GFH$ , the arc  $DE$  will also be equal to the arc  $GH$ ; and if the angles be not equal then the greater angle will stand upon the greater arc.

*Proof.* Place the circle  $B$  upon the circle  $A$  so that the point  $F$  falls upon the point  $C$ , and the bounding lines of the angle  $GFH$  upon those of the equal angle  $DCE$ ; then will the two circles coincide, and the points  $G$  and  $H$  will fall on the points  $D$  and  $E$  because the circles are of equal radii; (III. 1.)

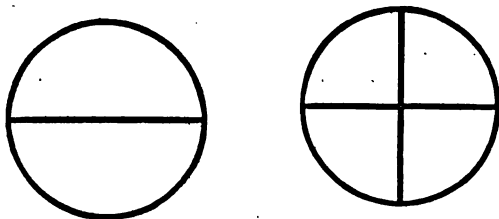
and the circumferences coinciding, and  $G$  and  $H$  falling on  $D$  and  $E$ , then will the arc  $GH$  fall on and coincide with the arc  $DE$ , and be therefore equal to it.

\* Euclid, III. 26.

Again, if the angles  $DCE$ ,  $GFH$  be not equal, let  $DCE$  be the greater. Then it is possible to place the circle  $B$  upon the circle  $A$  so that the point  $F$  falls on the point  $C$ , and the bounding lines of the angle  $GFH$  fall within the bounding lines of the greater angle  $DCE$ ; and therefore the minor arc  $GH$  forms a part of, and therefore is less than, the minor arc  $DE$ .

And further, since any angle  $KCL$  at the centre of the circle  $A$ , equal to  $GFH$ , stands upon an arc equal to  $GH$ , therefore the arc  $DE$  is greater than, equal to, or less than the arc  $KL$  in the same circle, according as the angle  $DCE$  is greater than, equal to, or less than the angle  $KCL$ .

**COR. 1.** *Sectors of the same or of equal circles which have equal angles are equal; and of two such sectors which have unequal angles that which has the greater angle is the greater.*



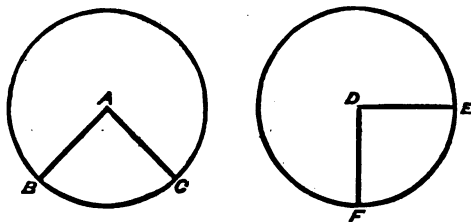
**COR. 2.** *A diameter of a circle divides it into two equal parts which are called semicircles, and two diameters at right angles to one another divide the circle into four equal parts which are called quadrants.*

*Proof.* The quadrants are sectors whose angles are right angles and are therefore equal to one another, by Cor. 1; and the semicircles are sectors whose angles are angles of two right angles, and are therefore equal to one another.

*Def. 7.* The former are called *semicircles*; and the latter are called *quadrants*.

## THEOREM 3.

*In the same circle, or in equal circles, equal arcs subtend equal angles at the centre; and of two unequal arcs the greater subtends the greater angle at the centre\*.*



*Part. En.* Let the circles whose centres are  $A$  and  $D$  be equal; and let  $BC$ ,  $FE$  be arcs;

then it is required to prove that if the arc  $BC$  be equal to the arc  $FE$ , the angle  $BAC$  will be equal to the angle  $FDE$ ; and if the arc  $BC$  be greater than the arc  $FE$ , the angle  $BAC$  will be greater than the angle  $FDE$ .

*Proof.* First, let the arc  $BC$  be equal to the arc  $FE$ ; then the angle  $BAC$  will be equal to the angle  $FDE$ .

For the angle  $BAC$  is either equal to the angle  $FDE$  or unequal to it;

\* Euclid, III. 27.



but if the angle  $BAC$  were unequal to the angle  $FDE$ , then the arc  $BC$  would be unequal to the arc  $FE$ ; (III. 2.)

but it is not;

and therefore the angle  $BAC$  is equal to the angle  $FDE$ .

Secondly, let the arc  $BC$  be greater than the arc  $FE$ ; then the angle  $BAC$  will be greater than the angle  $FDE$ .

For the angle  $BAC$  is either greater than, equal to, or less than the angle  $FDE$ ;

But the angle  $BAC$  is not equal to the angle  $FDE$ ; for then the arc  $BC$  would be equal to the arc  $FE$ ; (III. 2.) but it is not;

nor is the angle  $BAC$  less than the angle  $FDE$ ;

for then the arc  $BC$  would be less than the arc  $FE$ ; (III. 2.)

but it is not;

therefore the angle  $BAC$  is greater than the angle  $FDE$ .

*Obs.* (1) This theorem affords an excellent example of an application of the *rule of conversion* (p. 3). It must be observed that

Theorem 2 forms in fact a group of theorems in which it is demonstrated that (see figure to Theorem 3),

(1) if the angle  $A$  is greater than the angle  $D$ , the arc  $BC$  is greater than the arc  $FE$ ;

(2) if the angle  $A$  is equal to the angle  $D$ , the arc  $BC$  is equal to the arc  $FE$ ;

(3) if the angle  $A$  is less than the angle  $D$ , the arc  $BC$  is less than the arc  $FE$ .

*Now of the Hypotheses of these theorems one must be true;*

for the angle  $A$  must be greater than, equal to, or less than the angle  $D$ ;

and of the Conclusions no two can be true at the same time; for the arc  $BC$  cannot be both greater than, and equal to, the arc  $FE$ ; therefore the *rule of conversion* applies to this group of Theorems; that is, their *converses*, which form Theorem 3, are true. In the text this proof is given in detail. But when these conditions are true of a group of theorems their converses are always necessarily true; and no special proof is necessary.

*Obs.* (2) This theorem might also be proved by direct superposition.

**COR.** *Equal sectors of the same or of equal circles have equal angles, and of two unequal sectors the greater has the greater angle.*

This may be proved as in the theorem, or by the rule of conversion, (which applies to the group of Theorems contained in Theorem 2, Cor.1), or by direct superposition.

#### EXERCISES ON SECTION I.

1. What arc of a circle is equal to its conjugate arc?  
What arc is half its conjugate arc?
2. What segment of a circle is also a sector of a circle?
3. Prove that if two circles cut one another they cannot have the same centre.
4. Divide a circle into eight equal parts by radii.
5. If there be two sectors of equal circles, and the angle of the first is any multiple of the angle of the second, prove that the area of the first is the same multiple of the area of the second.

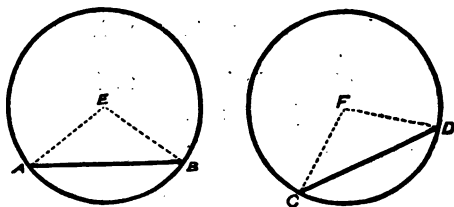
## SECTION II.

### CHORDS.

#### THEOREM 4.

*In the same circle, or in equal circles, equal arcs are subtended by equal chords; and of two unequal minor arcs, the greater is subtended by the greater chord\**

*Part. En.* Let  $AB$ ,  $CD$  be equal arcs of the same or of equal circles;



it is required to prove that the chord  $AB$  is equal to the chord  $CD$ .

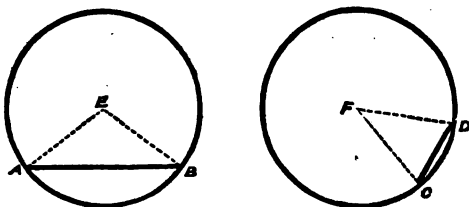
*Proof.* Let  $E$ ,  $F$  be the centres of the circles: join  $AE$ ,  $BE$ ,  $CF$ ,  $DF$ .

Then because the arc  $AB$  is equal to the arc  $CD$ ; (Hyp.) therefore the angle  $AEB$  is equal to the angle  $CFD$ ; (III. 3.) and because in the two triangles  $AEB$ ,  $CFD$ , the two sides

\* Euclid, III. 29.

$AE, EB$  are equal to the two  $CF, FD$ , and the contained angle  $AEB$  is equal to the contained angle  $CFD$ ; therefore the base  $AB$  is equal to the base  $CD$ . (I. 5.)

*Part. En.* Again, let the minor arc  $AB$  be greater than the minor arc  $CD$ ; it is required to prove that the chord  $AB$  is greater than the chord  $CD$ .



*Proof.* Because the arc  $AB$  is greater than the arc  $CD$ , therefore the angle  $AEB$  is greater than the angle  $CFD$ ;

(III. 3.)

and because in the triangles  $AEB, CFD$ , the two sides  $AE, EB$  are equal to the two  $CF, FD$ , but the contained angle  $AEB$  is greater than the contained angle  $CFD$ ;

therefore the base  $AB$  is greater than the base  $CD$ . (I. 14.)

*COR.* In the same circle, or in equal circles, of two unequal major arcs the greater is subtended by the less chord.

*Obs.* Since  $AB, CD$  are minor arcs, the angles  $AEB, CFD$  are less than two right angles. This has been assumed in the proof in treating  $AEB, CFD$  as triangles.

#### THEOREM 5.

*In the same circle, or in equal circles, equal chords subtend equal major and equal minor arcs; and of two unequal chords the greater subtends the greater minor arc and the less major arc\*.*

\* Euclid, III. 28.

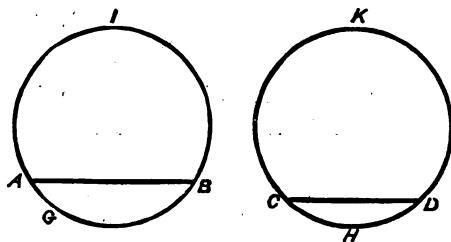
*Part. En.* Let  $AB$ ,  $CD$  be equal chords in the same or equal circles, it is required to prove that the arcs  $AB$ ,  $CD$  are equal.

*Proof.* For if the minor arcs  $AB$ ,  $CD$  were unequal, one of them would be the greater; and therefore the chord  $AB$  would be unequal to the chord  $CD$ . (III. 4.)

But it is not; for the chords are equal. (Hyp.)

Therefore the minor arcs  $AB$ ,  $CD$  are also equal.

Again, let the chord  $AB$  be greater than the chord  $CD$ ;



it is required to prove that the minor arc  $AGB$  is greater than the minor arc  $CHD$ .

*Proof.* For the minor arc  $AGB$  is either greater than, equal to, or less than the minor arc  $CHD$ ;

But the minor arc  $AGB$  is not equal to the minor arc  $CHD$ ,

for then the chord  $AB$  would be equal to the chord  $CD$ .

(III. 4.)

But it is not;

Nor is the minor arc  $AGB$  less than the minor arc  $CHD$ , for then the chord  $AB$  would be less than the chord  $CD$ .

(III. 4.)

But it is not;  
 therefore the minor arc  $AGB$  is greater than the minor arc  $CHD$ ,  
 and therefore also the major arc  $AIB$  is less than the major arc  $CKD$ .

*Obs.* As before, the rule of conversion applies to the groups of theorems enunciated in Theorem 4 and Cor., and their converses form Theorem 5.

## THEOREM 6.

*The straight line drawn from the centre of a circle to the middle point of a chord is perpendicular to the chord.*

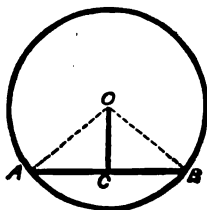
*Part. En.* Let the straight line  $OC$  be drawn from the centre  $O$  of a circle to the middle point  $C$  of a chord  $AB$ ; it is required to prove that  $OC$  is perpendicular to  $AB$ .

*Proof.* Join  $OA$ ,  $OB$ .

Then because in the triangles  $OAC$ ,  $OBC$ , the three sides of the one are respectively equal to the three sides of the other;

therefore the angles  $OCA$ ,  $OCB$  opposite to the equal sides  $OA$ ,  $OB$  are equal (I. 15), and are therefore right angles,

that is,  $OC$  is perpendicular to  $AB$ .\*



## THEOREM 7.

*The straight line drawn from the centre of a circle perpendicular to a chord bisects the chord†.*

\* Euclid, III. 3, Part 1.

† Euclid, III. 3, Part 2.

*Part. En.* Let the straight line  $OC$  drawn from the centre  $O$  of a circle be perpendicular to the chord  $AB$ ; it is required to prove that  $OC$  bisects  $AB$ .

*Proof.* Because in the right-angled triangles  $ACO$ ,  $BCO$ , the hypotenuse  $AO$  is equal to the hypotenuse  $BO$ , and the side  $OC$  is common, therefore the triangles are equal in all respects, (I. 20, Cor.) and therefore  $AC$  is equal to  $CB$ , and  $OC$  bisects  $AB$ .

#### THEOREM 8.

*The straight line drawn perpendicular to a chord of a circle through its middle point passes through the centre of the circle\*.*

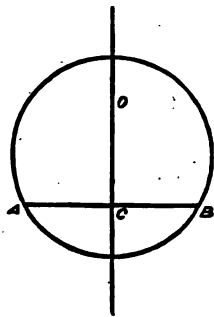
*Part. En.* Let  $AB$  be a chord of a circle, bisected in  $C$ , and let  $CO$  be drawn at right angles to  $AB$ ; it is required to prove that  $CO$  passes through the centre.

*Proof.* Because  $CO$  bisects  $AB$  at right angles; (Hyp.) therefore  $CO$  is the locus of points equidistant from  $A$  and  $B$  (p. 72). But the centre of the circle is equidistant from  $A$  and  $B$ :

therefore  $CO$  passes through the centre.

*COR.* *The locus of the centres of all circles that pass through two given points is the straight line that bisects at right angles the line joining those points.*

\* Euclid, III. 1. Cor.

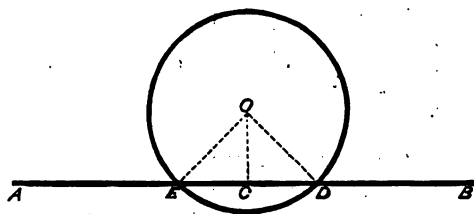


*Remark.* On Theorems 6, 7, 8 it may be remarked that if any one of them be proved directly, the other two follow from applications of the Rule of Identity. For example, if Theorem 7 be proved directly and it be required to demonstrate Theorem 6, we may proceed as follows:—Of straight lines through the centre there can be *but one* to the middle point of the chord, and *but one* perpendicular to it; and inasmuch as by Theorem 7, the one that is perpendicular to the chord is also the one that bisects it, it follows by the Rule of Identity that the one that bisects it is also the one that is perpendicular to it.

## THEOREM 9.

*A straight line cannot meet the circumference of a circle in more than two points.*

*Part. En.* Let  $AB$  be a straight line,  $O$  the centre of a circle;



It is required to prove that the straight line  $AB$  does not meet the circumference of the circle in more than two points.

*Proof.* Draw  $OC$  perpendicular to  $AB$ .

Let  $D$  be one of the points of intersection of the straight line and circle; join  $OD$ ;

and at the point  $O$  in the line  $OC$  on the side of  $OC$  remote from  $D$ , make the angle  $COE$  equal to  $COD$ , and let  $OE$  meet the straight line  $AB$  in  $E$ .



Then since  $OE$ ,  $OD$  are obliques making equal angles with the perpendicular  $OC$ ,

therefore  $OD$  is equal to  $OE$ , (I. 19.)

and therefore  $E$  is on the circumference of the circle,

and all lines drawn from  $O$  to  $AB$  other than  $OE$  are greater or less than  $OD$ ; that is no point on  $AB$  other than  $D$  and  $E$  is on the circle.

**COR.** *A chord of a circle lies wholly within the circle.*

**Obs.** Hence a circle is everywhere concave to its centre.

For the test of the concavity of an arc of a curve to a given point is that, if any two points in the arc be taken, the chord joining those points shall be cut by every line drawn from the given point to a point on that arc.



Thus in the figure the arc  $AB$  is concave to  $O$ , and the arc  $BC$  is not concave to  $O$ , and is said to be *convex* to it.

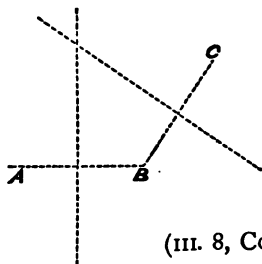
#### THEOREM 10.

*One circle, and only one, can be drawn through any three points not in the same straight line.*

**Part. En.** Let  $A$ ,  $B$ ,  $C$  be three points, not in the same straight line;

it is required to prove that one circle, and only one, can be drawn to pass through  $A$ ,  $B$ ,  $C$ .

*Proof.* Because the locus of the centres of all circles that pass through  $A$  and  $B$  is a straight line that bisects  $AB$  at right angles;



and the locus of the centres of all circles that pass through  $B$  and  $C$  is the straight line that bisects  $BC$  at right angles; therefore the centre of a circle that passes through  $A$ ,  $B$  and  $C$  is a point common to these two straight lines.

Now these straight lines will intersect, for if they did not they would be parallel, and therefore  $A$ ,  $B$ , and  $C$  would be in one straight line;

and they can intersect in only one point: (Ax. 2.)

therefore one circle, and only one circle, can be drawn to pass through  $A$ ,  $B$  and  $C$ .

**COR. 1.** *Two circles that have three points in common coincide wholly.*

Hence a circle is named by three letters at points on its circumference.

**COR. 2.** *Two circles cannot meet one another in more than two points\*.*

For if they had three points in common they would coincide wholly.

**COR. 3.** *If from any point within a circle more than two equal straight lines are drawn to the circumference, that point is the centre of the circle†.*

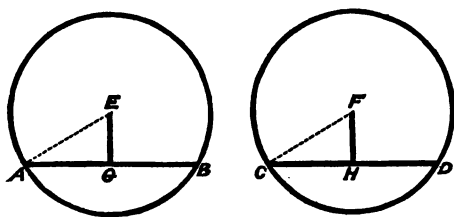
\* Euclid, III. 10.

† Euclid, III. 9.

## THEOREM II.

*In the same circle, or in equal circles, equal chords are equally distant from the centre; and of two unequal chords the greater is nearer to the centre than the less\*.*

*First. Part. En.* Let  $AB$ ,  $CD$  be equal chords in the same, or in equal circles, whose centres are  $E$ ,  $F$ : and let



$EG$ ,  $FH$  be perpendiculars from  $E$ ,  $F$  on  $AB$ ,  $CD$  respectively; it is required to prove that  $EG$  is equal to  $FH$ .

*Proof.* Join  $EA$ ,  $FC$ .

Then because  $EG$ ,  $FH$  are perpendiculars on the chords from the centres;

therefore they bisect the chords: (III. 7.)

and, because the chords are equal, (Hyp.)

therefore  $AG$  is equal to  $CH$ :

and therefore in the right-angled triangles  $AGE$ ,  $CHF$ ,

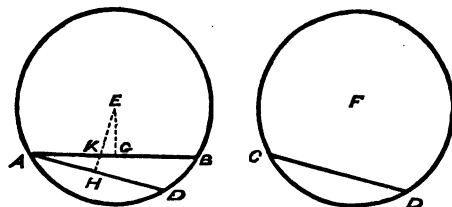
the hypotenuse  $AE$  is equal to the hypotenuse  $CF$ , (Hyp.)

\* Euclid, III. 14, 15. Part I.

and one side  $AG$  is equal to one side  $CH$ :  
 therefore the side  $EG$  is equal to  $FH$ : (I. 20, Cor.)  
 that is, the chords  $AB$ ,  $CD$  are equally distant from the centre.

Again,

*Part. En.* Let  $AB$ ,  $CD$  be unequal chords of the same, or of equal circles, of which  $AB$  is the greater;  
 it is required to prove that  $AB$  is nearer to the centre than  $CD$ .



*Proof.* Because the chord  $AB$  is greater than the chord  $CD$ ;

therefore the minor arc  $AB$  is greater than the minor arc  $CD$ . (III. 5.)

Place the circles on one another, with  $E$  as their common centre, (III. 1.)

so that the point  $C$  falls on  $A$ , and the point  $D$  on the minor arc  $AB$ ; and let fall the perpendiculars  $EG$ ,  $EH$  on the chords  $AB$ ,  $AD$ .

Let  $EH$  cut  $AB$  in  $K$ :

then because  $EG$  is the perpendicular on  $AB$  from  $E$ ;  
 therefore  $EG$  is less than  $EK$ : (I. 19.)

and  $EK$  is less than  $EH$ :

therefore  $EG$  is less than  $EH$ :

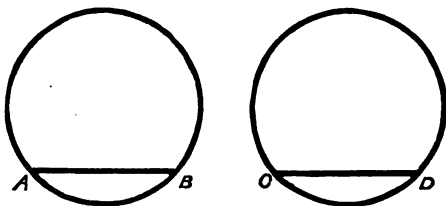
that is,  $AB$  is nearer to the centre than  $AD$  or  $CD$ .

*Obs.* The first part of this theorem may be proved by superposition.

#### THEOREM 12.

*In the same circle, or in equal circles, chords that are equally distant from the centre are equal; and of two chords unequally distant, the one nearer to the centre is the greater\*.*

*Part. En.* Let  $AB$ ,  $CD$  be chords of the same or of



equal circles, equally distant from the centre: it is required to prove that  $AB$  is equal to  $CD$ .

*Proof.* For if  $AB$  were unequal to  $CD$ , one of them would be the greater, and would therefore be nearer to the centre than the less. (III. 11.)

But it is not, for they are equally distant from the centre; (Hyp.)

therefore  $AB$  is equal to  $CD$ .

Again,

*Part. En.* Let  $AB$  be nearer to the centre than  $CD$ ; it is required to prove that  $AB$  is greater than  $CD$ .

\* Euclid, III. 14, 15. Part 2.

*Proof.* For  $AB$  must be either greater than  $CD$ , or equal to  $CD$ , or less than  $CD$ .

But  $AB$  is not equal to  $CD$ ;  
for then  $AB$  and  $CD$  would be equally distant from the centre. (III. 11.)

But they are not:

Nor is  $AB$  less than  $CD$ ;  
for then  $AB$  would be further from the centre than  $CD$ ;  
(III. 11.)

but it is not;  
therefore  $AB$  is greater than  $CD$ .

*COR.* The diameter is the greatest chord in a circle.

*Obs.* This theorem follows logically from Theorem 11. For the group of theorems in Theorem 11 is such that of their hypotheses *one must be true*, that is, the chord  $AB$  must be greater than, equal to, or less than the chord  $CD$ ; and of the conclusions,  $AB$  is at a less, equal, or greater distance from the centre than  $CD$ , *no two can be true at the same time*; therefore the rule of conversion is applicable; and Theorem 11 contains the converse theorems thus established.

It may also be remarked that this Theorem may be proved by superposition.

#### EXERCISES ON SECTION II.

1. Given a triangle  $ABC$  to find the centre of the circle that passes through  $A$ ,  $B$ , and  $C$ .
2. If two equal chords intersect one another, the segments of the one are equal to the segments of the other respectively.
3. Two chords cannot bisect one another unless both pass through the centre.

4. Given a curve, to ascertain whether it is an arc of a circle or not.

5. If a straight line cut two concentric circles, the parts of it intercepted between the two circumferences will be equal.

6. Perpendiculars are let fall from the extremities of a diameter on any chord, or any chord produced; shew that the feet of the perpendiculars are equally distant from the centre.

7. The locus of the points of bisection of parallel chords of a circle is the diameter at right angles to those chords.

8. If a diameter of a circle bisects a chord which does not pass through the centre, it will bisect all chords which are parallel to it.

9.  $AB$  and  $CD$  are unequal parallel chords in a circle; prove that  $AC$  and  $BD$ , and likewise  $AD$  and  $BC$ , intersect on the diameter perpendicular to  $AB$  and  $CD$ , or that diameter produced, and are equally inclined to that diameter.

What will be the case if  $AB$  and  $CD$  are equal?

## SECTION III.

### ANGLES IN SEGMENTS.

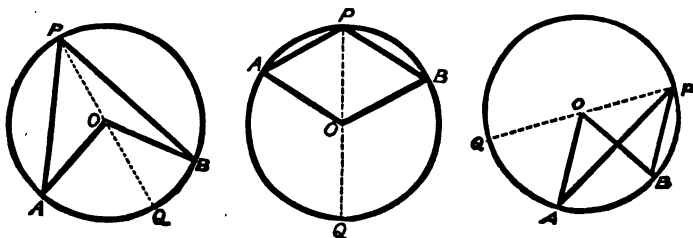
*Def. 8.* The angle formed by any two chords drawn from a point on the circumference of a circle is called an angle *at the circumference*, and is said to *stand upon the arc* between its arms.

*Def. 9.* An angle contained by two straight lines drawn from a point in the arc of a segment to the extremities of the chord is called an *angle in the segment*.

### THEOREM 13.

*An angle at the circumference of a circle is half the angle at the centre standing on the same arc.*

*Part. En.* Let  $AB$  be an arc,  $O$  the centre,  $P$  any





point on the circumference of a circle; it is required to prove that the angle  $APB$  is half of the angle  $AOB$  standing on the same arc.

*Proof.* Join  $PO$ , and produce it to  $Q$ .

Then because  $OA$  is equal to  $OP$ ;  
therefore the angle  $OAP$  is equal to the angle  $OPA$ : (I. 6.)  
but the exterior angle  $AOQ$  is equal to the two interior and opposite angles  $OAP$  and  $OPA$ ; (I. 25.)  
therefore the angle  $AOQ$  is double of the angle  $OPA$ .  
Similarly the angle  $QOB$  is double of the angle  $OPB$ .

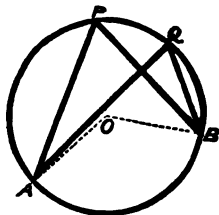
Hence in figs. (1 and 2) the sum, or (in fig. 3) the difference of the angles  $AOQ$ ,  $QOB$  is double of the sum or difference of  $OPA$  and  $OPB$ ,  
that is, the angle  $AOB$  is double of the angle  $APB$ ;  
and therefore the angle  $APB$  is half of the angle  $AOB$  on the same arc\*.

#### THEOREM 14.

*Angles in the same segment of a circle are equal to one another.*

*Part. En.* Let  $APB$ ,  $AQB$  be angles in the same segment  $APQB$ ;  
it is required to prove that the angle  $APB$  is equal to the angle  $AQB$ .

*Proof.* Take  $O$  the centre; and join  $AO$ ,  $BO$ .



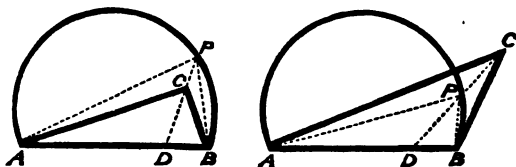
Then because the angles  $APB$ ,  $AQB$  at the circumference are each of them half the angle  $AOB$  at the centre on the same arc, (III. 13.)  
therefore the angle  $APB$  is equal to the angle  $AQB$ †.

\* Euclid, III. 20.

† Euclid, III. 21.

**COR.** *The angle subtended by the chord of a segment at a point within it is greater than, and at a point outside its segment on the same side of the chord as the segment, is less than, the angle in the segment.*

**Part. Ex.** Let  $APB$  be a segment of a circle, and  $C$  a point on the same side of  $AB$  as the segment; it is required to prove that the angle



$ACB$  is greater or less than the angle in the segment  $APB$  according as  $C$  is within or without the segment.

**Proof.** Take any point  $D$  in  $AB$  and join  $CD$ , and let  $CD$  (produced if necessary) meet the curved boundary of the segment in  $P$ . Join  $PA$ ,  $PB$ .

Then if  $C$  is within the segment  $APB$  it is evidently within the triangle  $APB$ , and therefore the angle  $ACB$  is greater than the angle  $APB$ . (I. 13.)

Again if  $C$  is without the segment  $APB$ ,  $P$  is evidently within the triangle  $ACB$ , and therefore the angle  $ACB$  is less than the angle  $APB$ . (I. 13.)

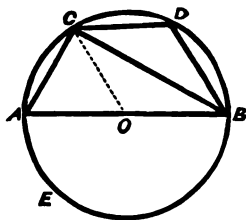
**Remark.** From this theorem and its corollary we learn that *the locus of a point on one side of a given straight line at which that straight line subtends a constant angle is an arc of a circle of which that line is the chord.*

#### THEOREM 15.

*The angle in a segment is greater than, equal to, or less than a right angle, according as the segment is less than, equal to, or greater than a semicircle\*.*

\* Euclid, III. 31.

*Part. En.* Let  $AB$  be a diameter of a circle, cutting off the semicircle  $ADB$ ; and let any other chord  $BC$  divide the circle into the segment  $BDC$  less than a semicircle, and  $BEC$  greater than a semicircle;



it is required to prove that the angle in the segment  $BDC$  less than a semicircle is greater than a right angle; and that the angle in the semicircle  $BDA$  is equal to a right angle; and that the angle in the segment  $BEC$  greater than a semicircle is less than a right angle.

*Proof.* Let  $O$  be the centre; join  $CO$ .

Then the angle in the segment  $CDB$  is half the angle  $COB$  subtended at the centre by the same arc  $BEC$ .

(III. 13.)

But this is a reflex angle, and is greater than two right angles;  
therefore the angle in the segment  $CDB$  is greater than one right angle.

Again, the angle in the semicircle  $ADB$  is half the angle  $AOB$  upon the same arc  $AEB$ .

(III. 13.)

But the angle  $AOB$  is equal to two right angles;  
therefore the angle in the semicircle is equal to a right angle.

Lastly, the angle in the segment  $CEB$  is half the angle  $COB$ .

(III. 13.)

But the angle  $COB$  is less than two right angles;

therefore the angle in the segment  $CEB$  is less than a right angle.

## THEOREM 16.

*A segment is less than, equal to, or greater than a semicircle according as the angle in it is greater than, equal to, or less than a right angle.*

*Proof.* According as the angle in the segment, that is at the circumference, is greater than, equal to, or less than a right angle; the angle at the centre will be greater than, equal to, or less than two right angles;

that is, the segment is less than, equal to, or greater than a semicircle.

*Alternative Proof.* For of the hypotheses that a segment is either greater than, equal to, or less than a semicircle, one must be true; and of the conclusions proved in Th. 15 that the angle in that segment is either less than, equal to, or greater than a right angle, no two can be true at the same time;

therefore the converses of the theorems in Th. 15 are true, that is; according as the angle in a segment is less than, equal to, or greater than a right angle, that segment is greater than, equal to, or less than a semicircle.

## THEOREM 17.

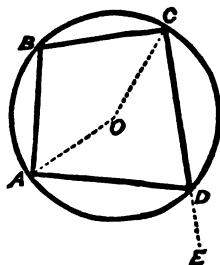
*The opposite angles of a quadrilateral inscribed in a circle are supplementary\*.*

\* Euclid, III. 22.

*Part. En.* Let  $ABCD$  be a quadrilateral inscribed in a circle; it is required to prove that its opposite angles  $ABC$ ,  $CDA$  are supplementary.

*Proof.* Take  $O$  the centre, and join  $AO$ ,  $CO$ .

Then the angles  $ABC$ ,  $CDA$  are respectively the halves of the angles made by  $AO$ ,  $OC$  at the centre  $O$ .  
(III. 13.)



But the sum of the angles at the centre  $O$  is four right angles :  
(I. 4, COR.)

therefore the sum of the angles  $ABC$ ,  $CDA$  is two right angles ;

that is, the angles  $ABC$ ,  $CDA$  are supplementary.

*COR. 1. The exterior angle of a quadrilateral inscribed in a circle is equal to the interior opposite angle.*

For if  $CD$  is produced to  $E$ , the exterior angle  $ADE$  is supplementary to  $ADC$ , and is therefore equal to  $ABC$ .

*COR. 2. If the opposite angles of a quadrilateral are supplementary, the quadrilateral can be inscribed in a circle.*

### EXERCISES ON SECTION III.

1. Prove that the lines which join the extremities of equal arcs in a circle are either equal or parallel.

2. If two opposite sides of a quadrilateral inscribed in a circle are equal, prove that the other two are parallel.

3.  $AB, CD$  are chords of a circle which cut at a constant angle. Prove that the sum of the arcs  $AC, BD$  remains constant, whatever may be the position of the chords.

4. If the diameter of a circle be one of the equal sides of an isosceles triangle, prove that its circumference will bisect the base of the triangle.

5. Circles are described on two sides of a triangle as diameters. Prove that they will intersect on the third side or third side produced.

6. Any number of chords of a circle are drawn through a point on its circumference: find the locus of their middle points.

7. If through any point, within or without a circle, lines are drawn to cut the circle, prove that the locus of the middle points of the chords so formed is a circle.

8. In any inscribed hexagon the sum of any three alternate angles is equal to four right angles.

## SECTION IV. A.

### TANGENTS (*treated directly*).

*Def. 10.* A secant is a straight line of unlimited length which meets the circumference of a circle in two points.

#### THEOREM 18.

*Every straight line through a point on the circumference of a circle meets it in one other point, except the straight line perpendicular to the radius at the point\*.*

*Part. En.* Let  $A$  be a circle,  $B$  its centre, and  $BC$  a radius; and let  $CD$  be a line through  $C$  perpendicular to the radius  $BC$ , and  $CE$  any other line;

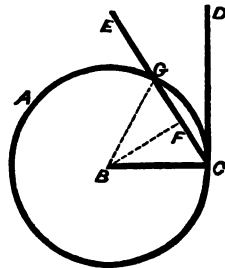
it is required to prove that  $CE$  meets the circle in one point other than  $C$ , and that  $CD$  does not.

*Proof.* Because  $BC$  is perpendicular to  $CD$ ,

therefore  $BC$  is the shortest line from  $B$  to the line  $CD$ : (I. 19.)

therefore every point in  $CD$  other than  $C$  is at a distance from  $B$  greater than  $BC$ , that is than the radius of the circle.

Therefore no point in  $CD$  other than  $C$  is on the circumference.



\* Euclid, III. 16.

Again, from  $B$  draw  $BF$  perpendicular to  $CE$ , and  $BG$  making an angle with  $BF$ , on the side remote from  $C$ , equal to  $CBF$ , and meeting  $CE$  in  $G$ .

Then because  $BC$  and  $BG$  are straight lines from  $B$  to the straight line  $CE$  making equal angles with the perpendicular  $BF$  upon it, they are equal; (I. 19.)

that is,  $BG$  is equal to the radius of the circle,

and therefore  $G$  lies upon the circumference; that is, the line  $CE$  meets the circle again in  $G$ .

*Def. 11.* A straight line which, though produced indefinitely, meets the circumference of a circle in one point only is said to *touch*, or to be a *tangent* to, the circle.

*Def. 12.* The point at which a tangent meets the circumference is called the *point of contact*.

The following are immediate consequences of Theorem 18.

(a) One and only one tangent can be drawn to a circle at a given point on the circumference.

(b) The tangent to a circle is perpendicular to the radius drawn to the point of contact.

(c) The centre of a circle lies in the perpendicular to the tangent at the point of contact.

(d) The straight line drawn from the centre perpendicular to the tangent passes through the point of contact.

*Obs.* On the relative position of a straight line and a circle.



A straight line will cut a circle, touch it, or not meet it at all, according as its distance from the centre is less than, equal to, or greater than the radius.

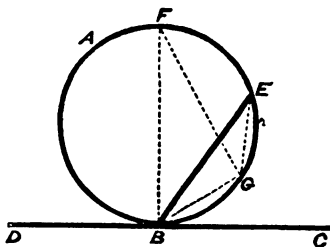
The several converses of these statements follow by the Rule of Conversion.

### THEOREM 19.

*Each angle contained by a tangent and a chord drawn from the point of contact is equal to the angle in the alternate segment of the circle\*.*

*Part. En.* Let  $DBC$  be a tangent to the circle  $A$  at the point  $B$ , and let  $BE$  be a line through  $B$  meeting the circle again in  $E$ ;

it is required to prove that the angles contained by  $DBC$  and  $BE$  are equal to the angles in the alternate segments upon  $BE$ .



*Proof.* Draw  $BF$  the diameter through  $B$ ; then  $BF$  will be at right angles to  $DC$ ; (Th. 18.) and join  $F, E$  and  $B$  to any point  $G$  in the minor arc  $BE$ .

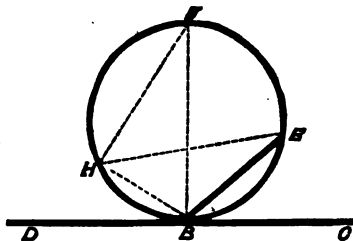
Then because  $FGB$  is an angle in a semicircle it is a right angle;

and therefore the angle  $FGB$  is equal to the angle  $FBD$ ; also the angle  $EGF$  is equal to the angle  $EBF$  in the same segment;

\* Eucl. III. 32.

therefore the whole angle  $EGB$  is equal to the whole angle  $EBD$ .

Again, join  $F$  and  $E$  and  $B$  to any point  $H$  in the circumference on the side of  $BF$  remote from  $E$ . Then because  $FHB$  is an angle in a semicircle it is a right angle;  
therefore the angle  $FHB$  is equal to the angle  $FBC$ ;



also the angle  $FHE$  is equal to the angle  $FBE$  in the same segment ;

therefore the remaining angle  $EHB$  is equal to the remaining angle  $EBC$ .

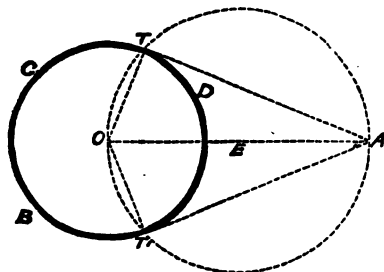
*Obs.* Having proved Th. 19 so far as it relates to *either* of the two angles  $EBC$ ,  $EBD$ , its truth as it relates to the *other* follows at once from Th. 17, since the angle in the conjugate segment and the remaining angle at  $B$  are respectively supplementary to the two equal angles.

#### THEOREM 20.

*Two tangents, and only two, can be drawn to a circle from an external point.*

*Part En.* Let  $A$  be a point external to the given circle  $BCD$ ; it is required to prove that two, and only two, straight lines can be drawn from  $A$  to touch the circle  $BCD$ .

*Proof.* Take  $O$  the centre ; join  $OA$ , bisect it in  $E$  ; and with centre  $E$  and radius  $EO$  or  $EA$  describe a circle.



Then  $OA$  will be a diameter of that circle, and each of the portions into which it divides the circumference will cut the circle  $BCD$ , because each is a continuous line with one extremity within and one extremity without the circle.

Let them meet it in  $T$  and  $T'$  respectively. Join  $OT$  and  $AT$ . Then because  $OTA$  is an angle in a semicircle it is a right angle,

(III. 15.)

therefore  $TA$  touches the circle  $BCD$  at the point  $T$ .

(III. 18.)

Similarly  $AT'$  touches the same circle at  $T'$ .

Therefore two straight lines can be drawn from  $A$  to touch the circle.

Again, there cannot be more than two straight lines drawn from  $A$  to touch the circle.

For because the angle between the radius and the tangent is a right angle,

(III. 18.)

therefore the point of contact lies on the circle described on  $AO$  as diameter.

(III. 16 and III. 13 Obs.)

But this circle cannot intersect the given circle in more than two points. (III. 10. Cor. 2.)

Therefore there cannot be drawn more than two tangents from  $A$  to the circle.

**COR.** *The two tangents drawn to a circle from an external point are equal and make equal angles with the straight line joining that point with the centre.*

For let  $AT, AT'$  be the two lines touching the circle in  $T$  and  $T'$ . Then because  $OT$  is equal to  $OT'$ , and  $OA$  is common to the two triangles  $OAT$  and  $OAT'$ , and the angles at  $T$  and  $T'$  are right angles; therefore the triangles are equal and the angle  $OAT$  is equal to the angle  $OAT'$ ; (I. 20.) and therefore the tangents from  $A$  are equal and make equal angles with  $OA$ .

## SECTION IV. B.

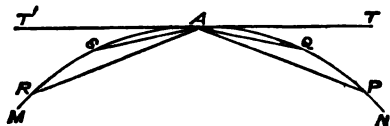
**TANGENTS** (*treated by the method of limits*).

*This may be omitted the first time of reading.*

There is another light in which we may regard the lines of which we have been speaking in Section IV. (A), which is extremely valuable when we come to consider curves other than circles. We shall proceed to give an account of it.

Let  $MAN$  be a curve, not necessarily a circle, but one which curves in the same direction throughout as you proceed from  $M$  towards  $N$ . Take a line through  $A$  meeting the curve at some point  $P$  between  $A$  and  $N$ . Then the

nearer  $P$  is taken to  $A$  the nearer does the line  $AP$  approach to a position represented in the figure by the



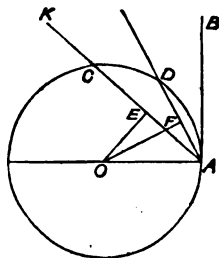
line  $T'AT$ . So long as  $P$  is between  $A$  and  $N$  it can never quite coincide with the said line  $TAT'$ , but it can be made to approach as near to it as we please by taking  $P$  close enough to  $A$ .

Similarly if we take a line through  $A$  meeting the curve in some point  $R$  between  $A$  and  $M$ , then the nearer that point lies to  $A$  the nearer will the line  $AR$  approach the position  $TAT'$ . It can never quite coincide with the said line, as long as  $R$  is between  $A$  and  $M$ , but it may be made to approach as near to it as we please by taking  $R$  near enough to  $A$ .

It may not be easy to see how the line  $T'AT$  is to be accurately obtained, but it will easily be seen that there is in general such a line at each point of a curve, and it will be distinguished from other lines drawn through the point by the peculiarity that *it does not cross the curve at that point*. Such a line is said to *touch* the curve at that point, or, more formally,—if a secant of a curve alters its position in such a manner that the two points of intersection continually approach, and ultimately coincide with one another, the secant in its limiting position is said to *touch*, or to be a *tangent* to, the curve, and the point at which the tangent meets the curve is called the *point of contact*.

We shall now investigate the position of the *tangent* to a circle at any specified point, using our newly obtained definition of a *tangent*.

Let  $A$  be the point and  $O$  the centre of the circle, and let  $AK$  be a straight line through  $A$ , not at right angles to  $OA$ , and  $AB$  a line through  $A$  perpendicular to  $OA$ . Draw  $OE$  perpendicular to  $AC$ , then  $OE < OA$ . (1. 19.)



$\therefore E$  is within the circle,

and  $AK$  must meet the circle in some point other than  $A$ . Let it be  $C$ .

Now let  $C$  move up towards  $A$ , then the chord  $AC$  will become shorter, and the perpendicular  $OE$ , which bisects the chord  $AC$ , will approach nearer to coincidence with  $OA$ . Hence the line  $AC$  will approach nearer to the position of being perpendicular to  $OA$ . And inasmuch as the chord  $AC$  can be made as short as we please, and thus the line  $OE$  can be made to approach as near as we please to  $OA$ , the line  $AK$  can be made to approach as near as we please to the position of  $AB$ . Hence  $AB$  is the *tangent* at  $A$ .

And inasmuch as no straight line can meet the circle in more than two points, and the line  $AB$  is the limiting position of a secant through  $A$  when the other point of intersection has moved up to coincidence with  $A$ , it follows that the line  $AB$  cannot meet the circle again. Hence every straight line through a point on the circumference meets it in one other point, except the straight line perpendicular to the

radius at the point, *and this is the tangent at the point*; which is Theorem 18.

It is evident that we shall arrive at exactly the same result by supposing that the point  $C$  is on the other side of  $OA$ , and moves up to coincidence with  $A$  in the other direction.

*Def. 11.* If a secant of a circle alters its position in such a manner that the two points of intersection continually approach, and ultimately coincide with one another, the secant in its limiting position is said to *touch* or to be a *tangent* to, the circle.

*Def. 12.* The point in which two points of intersection ultimately coincide is called the *point of contact* and the tangent is said to touch the circle at that point.

Taking this definition of a tangent, Theorem 6 gives us

*The straight line drawn from the centre to the point of contact of a tangent is perpendicular to the tangent.* This is (b) in the last section:

Theorem 7 gives us

*The straight line drawn from the centre perpendicular to a tangent passes through the point of contact.* This is (d) in the last section.

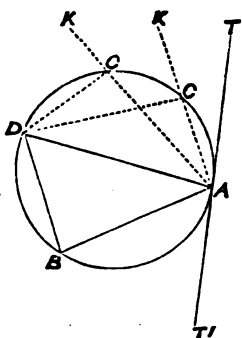
Theorem 8 gives us

*The straight line drawn perpendicular to a tangent through its point of contact passes through the centre.* This is (c) in the last section.

Theorem 17, Cor. 1, gives us Theorem 19.

For if  $ABDC$  be a quadrilateral inscribed in a circle and  $AC$  be produced to  $K$ , then the angle  $KCD$  is equal

to the angle  $ABD$ , that is, to the angle in the segment  $ABD$ . Now let  $C$  move up to coincidence with  $A$ . The angle  $DCK$  will remain the same in magnitude; and it finally coincides with  $DAT$ ; for  $AK$  will move up to coincidence with  $AT$  the tangent at  $A$ , and  $DC$  will move up to coincidence with  $DA$ . Hence the angle  $DAT$  is equal to the angle  $DBA$  in the alternate segment  $DBA$ . Similarly we can shew that  $DAT$  is equal to the angle  $DCA$  in the alternate segment  $DCA$ .



## EXERCISES ON SECTION IV.

1. Prove that the two tangents drawn to a circle from any external point are equal.
2. If from a point without a circle two tangents  $AB$ ,  $AC$  are drawn, the chord of contact  $BC$  will be bisected at right angles by the line from  $A$  to the centre.
3. If a circle is inscribed in a right-angled triangle, the excess of the two sides over the hypotenuse is equal to the diameter of the circle.
4. If a quadrilateral figure be described about a circle, the sums of the opposite sides will be equal to one another.
5. If a six-sided figure be circumscribed about a circle, the sums of the alternate sides will be equal.
6. If a quadrilateral figure be described about a circle, the angles subtended at the centre by any two opposite sides are together equal to two right angles.



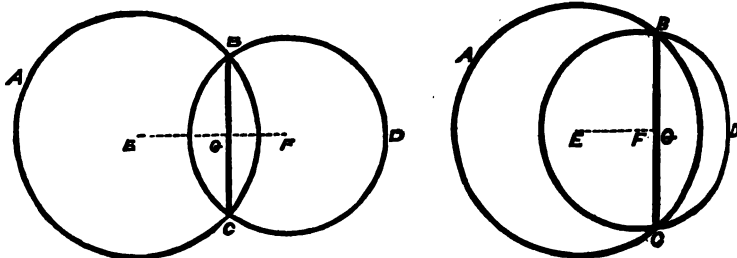
## SECTION V.

## THE RELATIONS OF TWO CIRCLES.

## THEOREM 21.

*The straight line which passes through the centres of two circles whose circumferences meet in two points bisects the straight line joining those points, and is at right angles to it.*

*Part. En.* Let  $ABC, DBC$  be two circles intersecting in the points  $B$  and  $C$ ; and let  $E$  and  $F$  be their centres;

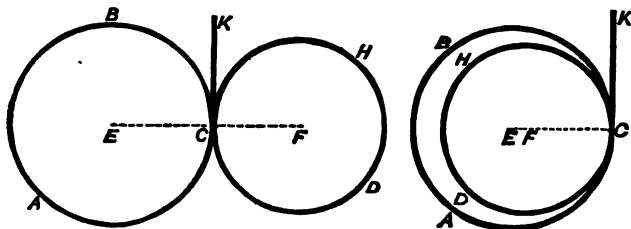


it is required to prove that the straight line  $EF$  bisects the common chord  $BC$  and is at right angles to it.

*Proof.* Bisect  $BC$  in  $G$ , and join  $GE$ ,  $GF$ . Then because  $BC$  is a chord of the circle  $BCD$ , and  $FG$  is a line drawn through the centre  $F$  bisecting it, therefore the angle  $BGF$  is a right angle; (III. 6.) and because  $BC$  is a chord of the circle  $BAC$ , and  $EG$  is drawn from the centre  $E$  bisecting it, therefore the angle  $EGB$  is a right angle; therefore  $EG$ ,  $GF$  are in one straight line, (I. 3.) that is, are in the straight line joining  $E$  and  $F$ . Therefore the straight line  $EF$  bisects the common chord  $BC$  and is at right angles to it.

## THEOREM 22.

*If the circumferences of two circles meet at a point on the straight line passing through their centres, these circumferences cannot have a second point in common.*



Let the two circles whose centres are  $E$ ,  $F$  have one point  $C$  in common on the straight line passing through their centres; then these circumferences shall not have any other point in common.

*Proof.* For if another point as  $B$  were common, then by Th. 21, the straight line  $EF$  would not pass through  $E$ , but bisect  $CB$  at right angles.

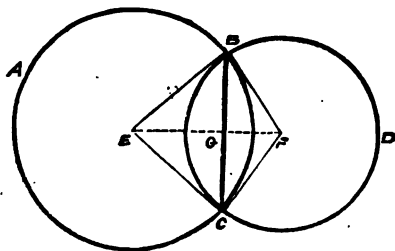
But  $EF$  does pass through  $C$ ;

therefore the circumferences have no other point in common except  $C$ .

*Def. 13.* Two circles whose circumferences meet in one point only are said to *touch* each other, and the point at which they meet is called their *point of contact*.

#### THEOREM 23.

*If the circumferences of two circles have one common point not on the line through their centres, they have also another common point.*



Let the two circles whose centres are  $E, F$  have one common point  $B$ , not on  $EF$ . They shall have also another common point.

*Proof.* From  $B$  let fall  $BG$  perpendicular to  $EF$ , and produce  $BG$  to  $C$ , making  $GC = GB$ . Join  $EB, EC, FB, FC$ .

Then because in the triangles  $EGC, EGB, CG = BG$  and  $EG$  in common, and the included angles are right angles, therefore  $EB = EC$ , and therefore  $C$  lies on the given circle whose centre is  $E$ .

In the same manner it may be proved that  $C$  lies on the given circle whose centre is  $B$ , that is, the circles have another common point  $C$ .

#### THEOREM 24.

*If two circles touch one another, the line through their centres passes through their point of contact\*.*

For if the point of contact were not in the line joining their centres, then the circles would have another common point, and therefore not touch one another.

*COR. Two circles that touch one another have a common tangent at the point of contact.* [By Theor. 18.]

*OBS. (1).* If the distance between the centres of two circles is greater than the sum of their radii, their circumferences will not meet and each circle will be wholly outside the other.

*OBS. (2).* If the distance between the centres of two circles is equal to the sum of their radii, their circumferences will meet in one point only, and each circle will lie outside the other.

\* Euclid, III. 11, 12.

*Def. 14.* In this case the circles are said to *touch externally*.

*OBS. (3).* If the distance between the centres of two circles is less than the sum and greater than the difference of their radii, their circumferences will meet in two points.

*Def. 15.* In this case the circles are said to *cut one another*.

*OBS. (4).* If the distance between the centres of two circles is equal to the difference of their radii, their circumferences will meet in one point only, and one circle will lie within the other.

*Def. 16.* In this case the circles are said to *touch internally*.

*OBS. (5).* If the distance between the centres of the two circles is less than the difference of their radii, their circumferences will not meet and one circle will be wholly within the other.

*OBS. (6).* The converse of each of the above five Theorems is true. [Rule of Conversion.]

### *Treatment by Limits.*

The results of Th. 22 may also be readily obtained from Th. 21 by means of the definition of Tangents given in Sect. IV. B. For if the two common points, which form the extremities of the common chord spoken of in Th. 21, move up to coincidence, the common chord becomes a common tangent, and the line joining the centres must be

perpendicular to the common tangent, and pass through its point of contact. In a similar way we may obtain Th. 23. For circles cannot meet in more than two points, and hence if these points move up to coincidence (so that the circles touch at that point) the circles can meet in no other point, and hence one must be wholly within the other or each must be wholly without the other.

We can also demonstrate by the method of limits an important proposition converse to Th. 23, viz. If two circles have but one common point they touch at that point. For it immediately follows

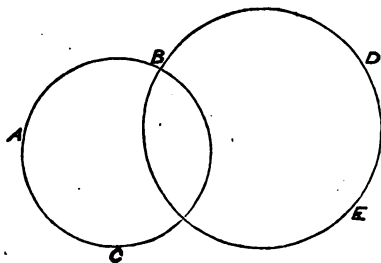
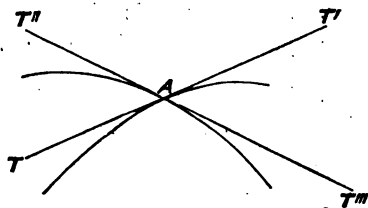
from the definition of a tangent given in Sect.

IV. B, that if two curves have a common point at which the tangents to the said curves make an angle with one another the curves must

cross at that point. But it is evident that if the circumferences of two circles  $ABC$  and  $DBE$  cross

at any point  $B$ , the circles must have another point common, for on one side of  $B$  the circumference of the circle  $ABC$  falls within the

other circle, and on the other side of  $B$  it is without the same; but circles are continuous curves, therefore the circumference of  $ABC$  must cross that of  $BED$  at some point



other than  $B$ . Hence if two circles have but one common point  $B$  they cannot cross one another there ; and therefore their tangents at  $B$  cannot include an angle but must coincide.

### EXERCISES.

1. If a straight line touch the inner of two concentric circles, and be terminated by the outer, prove that it will be bisected at the point of contact.

2. Any two chords which intersect on a diameter and make equal angles with it are equal.

3. Two fixed circles touch each other externally, and a third circle is described touching both externally. Shew that the difference of the distances of its centre from the centre of the two given circles will be constant.

4. If two circles intersect one another, and circles are drawn to touch both, prove that either the sum or the difference of the distances of their centres from the centres of the fixed circles will be constant, according as they touch (1) one internally and one externally, (2) both internally or both externally.

5. If two circles touch one another, any line through the point of contact will cut off segments from the two circles which contain the same angle.

6. If two circles touch one another, any two straight lines through the point of contact will cut off arcs, the chords of which are parallel.

7. Two circles cut one another, and lines are drawn through the points of section and terminated by the circumference, shew that they intercept arcs the chords of which are parallel.

8. Circles whose radii are 6·7 and 7·8 inches are successively placed so as to have their centres 14,  $14\frac{1}{2}$ , and 15 inches apart. Shew whether the circles will meet or touch or not meet one another.

9. What will be the case if the centres are 1 inch, 1·1 inch, or 1·2 inches apart?



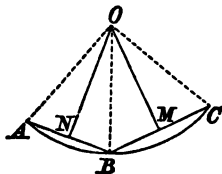
## SECTION VI.

## PROBLEMS.

## PROBLEM I.

*To find the centre of a given circle, or of a given arc\*.*

Let  $ABC$  be the arc.



*Construction.* Draw any two chords  $AB$ ,  $BC$ , and bisect them at right angles (Book I. Problems 2, 4) by straight lines  $ON$ ,  $OM$ , which will intersect at  $O$ .  $O$  shall be the centre required.

*Proof.* For  $NO$  is by construction the locus of points equidistant from  $A$  and  $B$ , and therefore  $AO = BO$ .

\* Eucl. III. 1.

Similarly,  $MO$  is the locus of points equidistant from  $B$  and  $C$ ; therefore  $O$  is equidistant from  $A$ ,  $B$  and  $C$ .

Hence, the circle described with centre  $O$  and radius equal to one of these three lines, will pass through the other two, and having three points coinciding with the given circular arc, must coincide with it throughout.

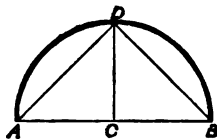
(III. 10. Cor. 1.)

### PROBLEM 2.

*To bisect a given arc\*.*

Let  $AB$  be the given arc; it is required to bisect it.

*Construction.* Join  $AB$ ; and bisect  $AB$  at  $C$ , and draw  $CD$  at right angles to  $AB$ , to meet the arc in  $D$ .



Then the arc  $AB$  is bisected in  $D$ .

*Proof.* Join  $AD$ ,  $BD$ . Then, since by construction,  $CD$  is the locus of points equidistant from  $A$  and  $B$ , therefore  $AD = BD$ .

But equal chords cut off equal arcs; (III. 5.)  
and therefore the arcs  $AD$ ,  $BD$  are equal.

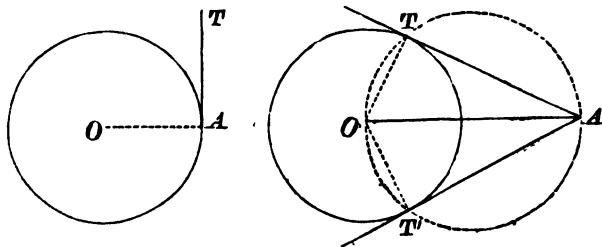
\* Euclid, III. 30.

## PROBLEM 3.

*To draw a tangent to a circle from a given point on or outside the circumference\*.*

There will be two cases.

First, let the given point  $A$  be on the circumference. Let  $O$  be the centre.



*Construction.* Join  $OA$ , and draw  $AT$  at right angles to  $OA$  (1. Prob. 2).

*Proof.* Then  $AT$  is a tangent. (Th. 18.)

Secondly, let  $A$  be outside the circle.

*Construction.* Join  $OA$ , and on it as diameter describe a circle, cutting the given circle in  $T$  and  $T'$ . Join  $AT$ ,  $AT'$ ; these shall be tangents from  $A$ .

*Proof.* Join  $OT$ ,  $OT'$ . Then since  $AFO$  is a semi-circle, the angle  $ATO$  is a right angle (III. 15). That is,  $AT$  or  $AT'$  is at right angles to the radius to the point where it meets the circumference, and therefore  $AT$  and  $AT'$  are tangents. (Th. 18.)

\* Euclid, III. 17.

## PROBLEM 4.

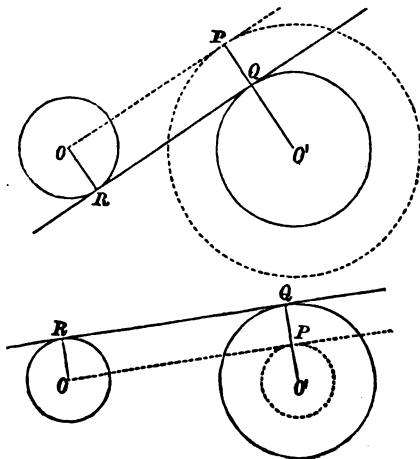
*To draw a common tangent to two given circles.*

Let the centres of the circles be  $O$ ,  $O'$ .

*Construction.* With centre  $O'$  and radius equal to the sum or difference of the radii of the given circles, describe a circle, as in the figures.

From  $O$  draw a tangent to this circle, touching it in  $P$  (III. Prob. 3). Join  $O'P$ , and let it, produced through  $P$  if necessary, meet the circumference of the circle whose centre is  $O'$  in the point  $Q$ . Through  $O$  draw  $OR$  parallel to  $PQ$  on the same side of  $OP$  as  $Q$  to meet the circle whose centre is  $O$  in  $R$ , and join  $QR$ .  $QR$  will be a tangent to both circles.

*Proof.* Since  $PQ$  is by construction equal and parallel to  $OR$ , therefore  $RQ$  is parallel to  $OP$ . (I. 30.)



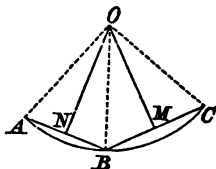
But  $OP$  is at right angles to  $O'P$ , since it touches the circle in  $P$ , and therefore  $RQ$  is at right angles to  $OQ$ ; and it is also at right angles to  $OR$ , therefore it touches both circles.

*COR. When the circles are wholly outside one another, they have four common tangents: when they touch externally, they have three common tangents: when they intersect one another, they have two common tangents: when they touch internally, they have one common tangent: and when one of the circles is wholly inside the other, they have no common tangent.*

#### PROBLEM 5.

*To describe a circle passing through three points which are not in the same straight line.*

Let  $A, B, C$  be the three points which are not in the same straight line. It is required to describe a circle to pass through  $A, B$  and  $C$ .



*Construction.* Join  $AB, BC$ . Bisect  $AB$  at right angles by the straight line  $NO$ , and bisect  $BC$  at right angles by the straight line  $MO$ , meeting the former in  $O$ . Then with

centre  $O$ , at the distance  $OA$ , describe a circle. It will pass through  $B$  and  $C$ .

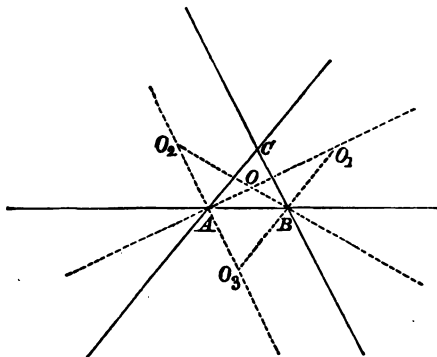
*Proof.* Because, by construction,  $NO$  is the locus of points equidistant from  $A$  and  $B$ ; therefore  $OA = OB$ . And because  $MO$  is the locus of points equidistant from  $B$  and  $C$ ; therefore  $BO = CO$ . Therefore the circle described with centre  $O$ , and radius  $OA$ , will pass through  $A$ ,  $B$  and  $C$ .

#### PROBLEM 6.

*To describe a circle to touch three given straight lines of indefinite length, which are not all parallel, and do not all pass through the same point.*

Let the three given lines intersect in  $A$ ,  $B$ , and  $C$ .

Then, since the circle required is to touch the lines that intersect in  $A$ , its centre must lie on one of the bisectors



of the angles at  $A$  (III. 20. Cor.). Similarly, it must lie on

one of the bisectors of the angles at  $B$ . Therefore the construction is as follows:

*Construction.* Draw the bisectors of the angles at  $A$  and  $B$ , which will intersect in four points  $O$ ,  $O_1$ ,  $O_2$ ,  $O_3$ .

These will be the centres of the circles required, and a circle described with any one of these points as centre, to touch one of the given lines, will touch the other two.

*COR. 1.* It follows that  $COO_3$  and  $O_2CO_1$  are straight lines, that is, the six bisectors of the interior and exterior angles of a triangle intersect one another three and three in four points.

*COR. 2.* If two of the lines are parallel, only two circles can be described to touch the three lines.

*COR. 3.* If all the lines are parallel, or if they all pass through one point, no circle can be described to touch them all.

*Def. 17.* A circle that touches the three sides of a triangle is called an *inscribed circle*.

*Def. 18.* A circle that touches one side of a triangle and the other two sides produced is called an *escribed circle*.

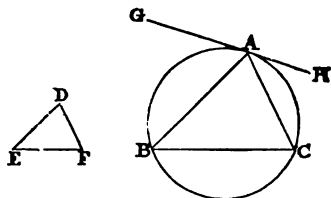
#### PROBLEM 7.

*In a given circle to inscribe a triangle equiangular to a given triangle.*

*Construction.* Let  $ABC$  be the given circle, and  $DEF$  the given triangle: it is required to inscribe in the circle  $ABC$  a triangle equiangular to the triangle  $DEF$ .

Draw the straight line  $GAH$  touching the circle at the point  $A$ ;

at the point  $A$ , in the straight line  $AH$ , make the angle  $HAC$  equal to the angle  $DEF$ ;



and, at the point  $A$ , in the straight line  $AG$ , make the angle  $GAB$  equal to the angle  $DFE$ ; and join  $BC$ .

$ABC$  shall be the triangle required.

*Proof.* Because  $GAH$  touches the circle  $ABC$ , and  $AC$  is drawn from the point of contact  $A$ ,

therefore the angle  $HAC$  is equal to the angle  $ABC$  in the alternate segment of the circle. (III. 19.)

But the angle  $HAC$  is equal to the angle  $DEF$ .

Therefore the angle  $ABC$  is equal to the angle  $DEF$ .

For the same reason the angle  $ACB$  is equal to the angle  $DFE$ .

Therefore the remaining angle  $BAC$  is equal to the remaining angle  $EDF$ .

Wherefore the triangle  $ABC$  is equiangular to the triangle  $DEF$ , and it is inscribed in the circle  $ABC^*$ .

#### PROBLEM 8.

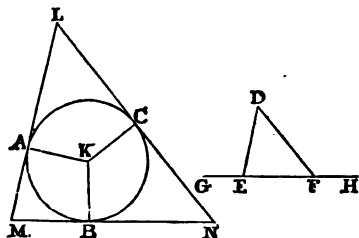
*About a given circle to circumscribe a triangle equiangular to a given triangle.*

\* Euclid, IV. 2.



Let  $ABC$  be the given circle, and  $DEF$  the given triangle: it is required to describe a triangle about the circle  $ABC$ , equiangular to the triangle  $DEF$ .

*Construction.* Produce  $EF$  both ways to the points  $G, H$ ; take  $K$  the centre of the circle  $ABC$ ;



from  $K$  draw any radius  $KB$ ;

at the point  $K$ , in the straight line  $KB$ , make the angle  $BKA$  equal to the angle  $DEG$ , and the angle  $BKC$  equal to the angle  $DFH$ ;

and through the points  $A, B, C$ , draw the straight lines  $LAM, MBN, NCL$ , touching the circle  $ABC$ :

$LMN$  shall be the triangle required.

*Proof.* Because  $LM, MN, NL$  touch the circle  $ABC$  at the points  $A, B, C$ ,

to which from the centre are drawn  $KA, KB, KC$ , therefore the angles at the points  $A, B, C$  are right angles.

And because the four angles of the quadrilateral figure  $AMBK$  are together equal to four right angles,

(for it can be divided into two triangles,)

and that two of them  $KAM, KBM$  are right angles,

therefore the other two  $AKB$ ,  $AMB$  are together equal to two right angles.

But the angles  $DEG$ ,  $DEF$  are together equal to two right angles.

Therefore the angles  $AKB$ ,  $AMB$  are equal to the angles  $DEG$ ,  $DEF$ ;

of which the angle  $AKB$  is equal to the angle  $DEG$ ;

therefore the remaining angle  $AMB$  is equal to the remaining angle  $DEF$ .

In the same manner the angle  $LMN$  may be shewn to be equal to the angle  $DFE$ .

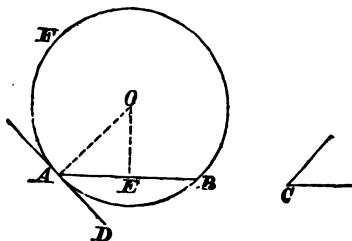
Therefore the remaining angle  $MLN$  is equal to the remaining angle  $EDF$ .

Wherefore the triangle  $LMN$  is equiangular to the triangle  $DEF$ , and it is described about the circle  $ABC^*$ .

#### PROBLEM 9.

*On a given straight line to describe a segment of a circle containing an angle equal to a given angle.*

Let  $AB$  be the given line,  $C$  the given angle.



\* Euclid, IV. 3.

*Construction.* At the point  $A$  make an angle  $BAD$  equal to the angle  $C$  (I. Prob. 6).

Then if a circle be described to touch  $AD$  in  $A$ , and to pass through  $B$ , the segment of that circle alternate to  $BAD$  will be the segment required. (Th. 19.)

To find the centre of this circle, draw  $AO$  at right angles to  $AD$ : then  $AO$  is the locus of the centres of all circles which touch  $AD$  at  $A$ . (III. 18. c.)

Bisect  $AB$  at right angles by the line  $EO$ ; then  $EO$  is the locus of the centres of circles which pass through  $A$  and  $B$ . (III. 8. Cor.)

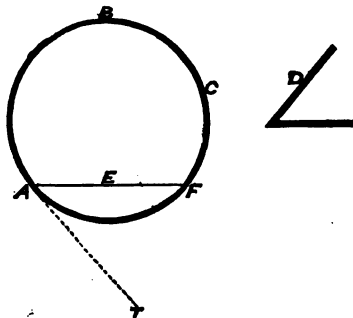
Therefore  $O$ , the point of intersection of these lines, is the centre of the circle required.

With centre  $O$  and radius  $OA$  or  $OB$  describe a circle, which will touch  $AD$  at  $A$  and pass through  $B$ , and therefore the segment  $AFB$  contains an angle equal to the alternate angle  $BAD$ , that is to the given angle  $C^*$ .

#### PROBLEM 10.

*From a given circle to cut off a segment containing a given angle.*

Let  $ABC$  be the given circle; it is required to cut off



\* Euclid, III 33.

from it a segment containing an angle equal to the given angle  $D$ .

*Construction.* At  $A$  any point on the circumference draw the tangent  $AT$ ; (III. Prob. 3.)

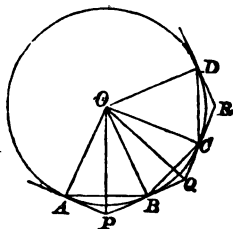
and at the point  $A$  in the straight line  $AT$  make the angle  $TAE$  equal to  $D$  (I. Prob. 6), and let  $AE$  meet the circle again in  $F$ ;

then the segment cut off by  $AF$ , remote from  $AT$ , shall contain an angle equal to the alternate angle  $FAT$  (III. 19), and therefore the segment  $ABF$  contains an angle equal to  $D^*$ .

### THEOREM 25.

*If the whole circumference of a circle is divided into any number of equal arcs, the inscribed polygon formed by the chords of these arcs is regular; and the circumscribed polygon formed by tangents drawn at all the points of division is also regular.*

*Part. En.* Let the circumference of the circle  $ABC$  be



\* Euclid, III. 34.

divided into any number of equal arcs in the points  $A, B, C, D, \dots$  it is required to prove that the polygon  $ABCD \dots$  is regular, and that so is also the polygon formed by tangents drawn at the points  $A, B, C, \dots$

*Proof.* Because the minor arcs  $AB, BC, CD \dots$  are all equal, the chords  $AB, BC, CD \dots$  are equal, and therefore the polygon  $ABCD \dots$  is equilateral. Also each of the angles  $ABC, BCD \dots$  stands upon an arc that is made up of all but two of the equal arcs into which the circle is divided; thus they stand upon equal arcs, and are therefore equal, and therefore the polygon is also equiangular.

It is therefore regular.

Again, draw tangents at the points  $A, B, C \dots$  and let them form the polygon  $PQR \dots$ . Take  $O$  the centre of the circle and join  $OA, OB$ .

Because the interior angles of the quadrilateral  $OAPB$  are equal to four right angles (I. 26.)

and those at  $A$  and  $B$  are right angles:

therefore the angles  $APB$  and  $AOB$  are together equal to two right angles, and  $APB$  is the supplement of  $AOB$ .

Similarly, each of the other angles of the circumscribing polygon is supplemental to one of the angles at the centre that stand upon the equal arcs into which the circle has been divided, and which are therefore equal.

Hence the polygon is equiangular.

Again, join  $OP, OQ$ . Then because the tangents from  $P$  make equal angles with the line  $PO$  to the centre of the circle, the angle  $OPB$  is one half the angle  $APB$ .

Similarly  $OQB$  is half the angle  $BQC$ , which has been shewn to be equal to  $APB$ .

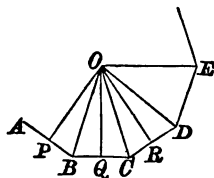
Therefore the angle  $OPB$  is equal to the angle  $OQB$ ; and therefore  $OP = OQ$ . (i. 8.)

Similarly it can be shewn that  $OQ = OR$ , and thus all the lines from  $O$  to the angular points of the circumscribing circle are equal, and a circle may be described with centre  $O$  passing through them all. Describe it; then since  $OA$ ,  $OB$ , and  $OC$  are all equal therefore the sides  $PQ$ ,  $QR$ ,... of the polygon are chords in it equally distant from the centre and are therefore equal. Hence the polygon formed by the tangents at  $A$ ,  $B$ ,  $C$ ,... is equilateral, and is therefore a regular polygon.

## THEOREM 26.

*If straight lines are drawn bisecting two angles of a regular polygon, the point in which the bisectors intersect is equidistant from all the vertices of the polygon and from all the sides.*

*Part. En.* Let  $ABCDE...$  be a regular polygon, and let  $BO$ ,  $CO$  be drawn bisecting the adjoining angles  $ABC$



and  $BCD$ . It is required to prove that the point  $O$  in which they meet is equidistant from all the angles and all

the sides of the polygon, and that it lies on all other bisectors of the angles of the polygon.

*Proof.* Join  $OD$ . Then because in the triangles  $OBC$ ,  $ODC$

$$BC = CD \quad (\text{Hyp.})$$

$CO$  is common,

$$\text{and } \angle BCO = \angle DCO; \quad (\text{Hyp.})$$

therefore the triangles are equal, and

the angle  $\angle CBO = \angle CDO$ ;

and therefore  $\angle CDO$  is equal to one half one of the angles of the regular polygon. Therefore the line  $OD$  bisects the angle  $\angle CDE$ .

And similarly we may shew that the line from  $O$  to each angular point of the polygon bisects that angle of the polygon.

Again, because  $\angle OCD$  and  $\angle ODC$  are each the half of an angle of the polygon, they are equal, and the side  $OC = OD$ .

Similarly each of the lines  $OA$ ,  $OB$ , and  $OC$  is equal to the next,

and therefore they are all equal,

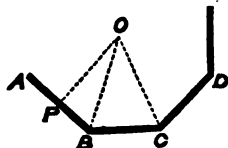
and a circle can be described with centre  $O$  passing through all the angular points of the polygon. Moreover the sides are equal chords in this circle, and are therefore equally distant from the centre  $O$ .

Hence  $O$  lies on the bisector of each angle of the polygon, and is equidistant from all its sides and angular points.

## PROBLEM II.

*To inscribe a circle in, or to circumscribe a circle about, a regular figure.*

Let  $ABCD\dots$  be a regular figure, it is required to in-



scribe a circle in it, and also to circumscribe a circle about it.

*Construction.* Bisect the two adjacent angles  $ABC$ ,  $BCD$  of the figure by the lines  $BO$ ,  $CO$  meeting in  $O$ . From  $O$  draw the perpendicular  $OP$  on  $AB$ . With centre  $O$  and radius  $OP$  describe a circle, it shall be inscribed in the figure  $ABCD\dots$ ; and with centre  $O$  and radius  $OB$  describe a circle, it shall be circumscribed about the said figure.

*Proof.* Because  $BO$  and  $CO$  bisect two adjacent angles of the regular figure  $ABCD\dots$  the point  $O$  where they meet is equidistant from all the sides of that figure. (Th. 26.)

Therefore the circle whose centre is  $O$  and radius  $OP$  will pass through the feet of all the perpendiculars from  $O$  upon the sides of the figure, and the said sides, being perpendicular to the radii drawn to the points where they meet the circle, will touch the circle; therefore the circle is inscribed in the regular figure  $ABCD\dots$ .

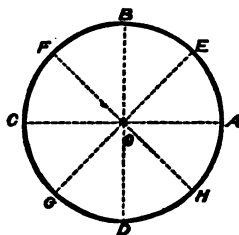
Similarly, because  $O$  is equidistant from all the angles of the polygon; (III. 26.)  
therefore the circle described with centre  $O$  and radius  $OB$  will be circumscribed to the polygon.



## PROBLEM 12.

*To inscribe in, or to circumscribe about, a given circle regular figures of 4, 8, 16, 32,... sides.*

*Construction.* Let  $O$  be the centre of the circle. Draw two diameters  $AOC$ ,  $BOD$  at right angles to one another.



Then because the four angles that they form at the centre are equal, the points  $A, B, C, D$  divide the circumference of the circle into four equal arcs.

Again, by bisecting each of the angles thus formed at the centre by lines meeting the circumference, we shall divide the circumference into 8 equal arcs, and by repeating the process we can divide it into 16, 32... equal parts.

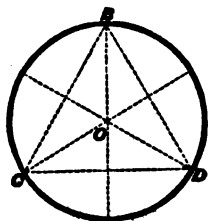
Then the chords of the equal arcs will form a regular inscribed figure of the prescribed number of sides, (III. 25.)

and the tangents at the points of division will form a regular circumscribed figure of the prescribed number of sides. (III. 25.)

## PROBLEM 13.

*To inscribe in, or to circumscribe about, a given circle regular figures of 3, 6, 12, 24... sides.*

*Construction.* Let  $O$  be the centre of the circle. Inscribe in the circle the equilateral triangle,  $BCD$  (III.



Prob. 7), and join  $OB$ ,  $OC$ ,  $OD$ . Then the angles  $BOC$ ,  $COD$ ,  $DOA$  are equal, (III. 5 and 3.)

and by bisecting them by lines meeting the circumference we shall divide the circumference into six equal arcs. Again, by bisecting the angles which the said arcs subtend at the centre we shall divide the circumference into 12 equal arcs, and by repeating the process we shall divide it into 24, 48... equal parts.

Then the chords of the equal arcs will form a regular inscribed figure of the prescribed number of sides. (III. 25.)

And the tangents at the points of division will form a regular circumscribed figure of the prescribed number of sides. (III. 25.)

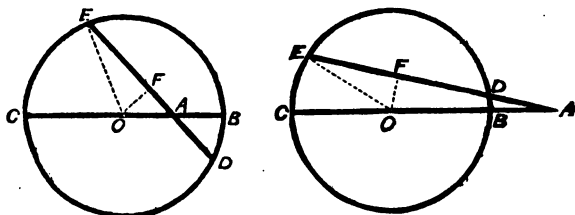
## SECTION VII.

## THE CIRCLE IN CONNECTION WITH AREAS.

## THEOREM 27.

*If a chord of a circle is divided into two segments by a point in the chord or in the chord produced, the rectangle contained by these segments is equal to the difference of the squares on the radius and on the line joining the given point with the centre of the circle.*

*Part. En.* Let  $A$  be a point and  $CEB$  a circle, whose centre is  $O$ . Then the rectangle contained by the seg-



ments into which  $A$  divides any chord through it, shall be equal to the difference between the squares on  $OA$  and on the radius.

*Proof.* Join  $AO$  and let it cut the circle in  $B$  and  $C$ , and draw through  $A$  any other chord  $DE$  not at right angles to  $AO$ . Draw  $OF$  the perpendicular from  $O$  upon it and join  $OE$ .

Then because the square on  $AO$  is equal to the squares on  $OF$ ,  $FA$ , (II. 9.)

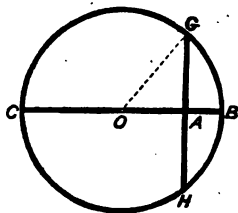
and the square on  $OE$  is equal to the squares on  $OF$ ,  $FE$ ; therefore the difference between the squares on  $OA$  and  $OE$  is equal to the difference between the squares on  $AF$ ,  $FE$ ,

that is, to the rectangle contained by the sum and difference of  $AF$  and  $FE$ . (II. 8.)

But  $AE$  is the sum of  $AF$ ,  $FE$ , and  $AD$  is the difference between  $AF$ ,  $FE$ , since  $FE = FD$ ; (III. 7.)

therefore the rectangle contained by  $AE$ ,  $AD$  is equal to the difference of the squares on  $AO$  and the radius.

Again, if  $A$  be within the circle and  $GAH$  be the chord bisected at  $A$  and  $OG$  be joined, it is obvious that because  $GAO$  is a right angle the difference between the squares of  $OA$  and the radius is equal to the square of  $AG$ , that is, to the rectangle under  $GA$ ,  $AH$ .



Therefore if any chord be drawn through  $A$  the rectangle under the segments into which it is divided internally or externally by  $A$  is equal to the difference between the squares of the radius and the distance of  $A$  from the centre of the circle.

COR. 1. *The rectangle contained by the segments of any chord of a circle passing through a given point is the same, whatever be the direction of the chord\*.*

COR. 2. *If the point is within the circle, the rectangle contained by the segments of any chord passing through it is equal to the square on half that chord which is bisected by the given point.*

COR. 3. *If the point is without the circle, the rectangle contained by the segments of any chord passing through it is equal to the square on the tangent to the circle drawn from that point†.*

For if  $OT$  be the tangent its square is equal to the difference between the squares of  $OA$  and the radius, since the angle  $OTA$  is a right angle.

COR. 4. *Conversely, if the rectangle contained by the segments of a chord passing through an external point is equal to the square of a line joining that point with a point in the circumference of the circle, this line touches the circle‡.*

For by the last corollary it must be equal in length to each of the two tangents from the point, and therefore must be one of them, since by Theorem 10, Cor. 3, there cannot be more than two equal straight lines drawn to the circle from a point not the centre.

\* Euclid, III. 35.

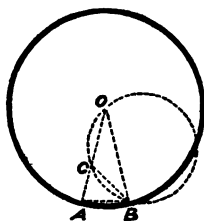
† Euclid, III. 36.

‡ Euclid, III. 37.

## PROBLEM 14.

*To inscribe in a circle a regular decagon; and thence to circumscribe a regular decagon about a circle; also to inscribe in, or to circumscribe about, a given circle a regular pentagon, or regular figures of 20, 40, 80... sides.*

*Construction.* Let  $O$  be the centre of the circle. Take any radius  $OA$  and divide it in  $C$  so that the rectangle



under  $OA$  and  $AC$  is equal to the square on  $OC$ .

(II. Prob. 5.)

Draw a chord  $AB$  of the circle equal to  $OC$ . It shall cut off an arc equal to one-tenth part of the whole circumference.

*Proof.* Join  $OB$  and  $CB$ , and describe a circle round the triangle  $OBC$ . Then because  $AB$  is equal to  $OC$ , the rectangle under  $OA$  and  $AC$  is equal to the square of  $AB$ .  
(Constr.)

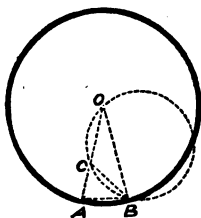
And because the rectangle under  $OA$ ,  $AC$  the segments of a chord of the circle  $OBC$  drawn from the external point  $A$ , is equal to the square of the line joining  $A$  with a point  $B$  on that circle, therefore  $AB$  touches that circle;

(III. 27, Cor. 4.)

and  $BC$  is a chord drawn from the point of contact;

therefore the angle  $CBA$  is equal to the angle  $BOC$  in the alternate segment  
(III. 19.)

and the angle  $BAC$  is common to the two triangles  $BCA, AOB$ ,



therefore the third angle  $ACB$  of the one is equal to the third angle  $ABO$  of the other:

but the angle  $ABO$  is equal to the angle  $BAO$ , because  $OA$  is equal to  $OB$ ; (I. 6.)

therefore the angle  $ACB$  is equal to the angle  $BAC$ , and therefore the side  $BC$  is equal to the side  $AB$ , and therefore to  $CO$ ; (Constr.)

therefore  $OBC$  is an isosceles triangle, and the exterior angle  $ACB$  at the vertex is double of the angle  $BOC$ , one of the equal angles at the base; (I. 25.)

but  $OAB$  and  $OBA$  are each of them equal to  $ACB$ , therefore they are each double the angle  $AOB$ , which is the remaining angle of the triangle  $AOB$ ;

therefore the angle  $AOB$  is one-fifth part of the sum of the angles of the triangle  $AOB$ , that is of two right angles;

therefore it is one-tenth part of four right angles;

therefore the arc  $AB$  on which it stands is one-tenth part of the whole circumference.

We can thus divide the circumference into ten equal parts, and so into 20, 40, &c. by bisecting the said equal

parts or the angles they subtend at the centre, also by taking every alternate point of division we can divide the circumference into five equal parts,

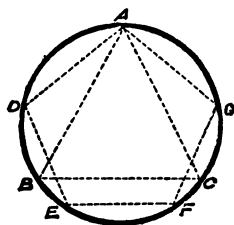
and thus by joining the points of division, or drawing tangents to the circle thereat, we can inscribe and circumscribe about the circle regular figures of 5, 10, 20... sides\*.

(III. 25.)

#### PROBLEM 15.

*To inscribe in a circle a regular quindecagon; and thence to circumscribe a regular quindecagon about a circle; also to inscribe in, or to circumscribe about, a given circle regular figures of 30, 60, 120... sides.*

*Construction.* Let  $AD$ ,  $DE$  be sides of a regular pentagon inscribed in the circle, and  $AB$  the side of an equilateral



triangle inscribed in the circle. Then  $BE$  shall be the fifteenth part of the circumference.

*Proof.* Because  $AE$  is two-fifths of the circumference and  $AB$  is one-third, therefore  $BE$  is one-fifteenth part of the circumference,

and by proceeding as in Problems 12, 13, and 14 we can circumscribe a regular quindecagon about a circle, and also inscribe in, or circumscribe about, a given circle regular figures of 30, 60, 120... sides†.

\* Euclid, IV. 10.

† Euclid, IV. 16.



*Remark.* Regular polygons can therefore be constructed when the number of their sides is 3, 4, 5, or 15, or these numbers multiplied by any power of 2. And besides these no other regular polygons can be constructed by the use of the ruler and compasses only, with the remarkable exception discovered by Gauss; who shewed that a polygon of  $2^n + 1$  sides can be described by the ruler and compasses alone, when  $n$  is such that  $2^n + 1$  is a prime number. If  $n$  has the values 1, 2, 3... in succession,  $2^n + 1$  takes the values 3, 5, 9, 17, 33, 65, 129, 257... of which 3, 5, 17, 257 are primes. Hence Gauss has shewn that regular polygons of 17 and 257 sides *can* be constructed by the use of the ruler and compasses; but the construction and proof, even for the first of these, are far too tedious to be given in an elementary work.

#### EXAMPLES ON BOOK III.

1. Two circles touch one another in  $A$ , and have a common tangent  $BC$ . Shew that the angle  $BAC$  is a right angle.
2.  $AOB$ ,  $COD$  are two chords of a circle at right angles to one another; prove that the squares of  $OA$ ,  $OB$ ,  $OC$ , and  $OD$ , are together equal to the square of the diameter.
3. With the same hypothesis, if  $M$  is the centre, prove that  $AB^2 + CD^2 = 8AM^2 - 4OM^2$ .
4. Describe a circle to touch a given line in a given point, and pass through another given point.
5. Describe a circle to touch a given circle in a given point, and to pass through another given point.
6. Find the locus of the point of intersection of the lines which bisect the angles at the base of triangles on the same base and having a given vertical angle.  
(Prove that the angle between each pair of bisectors is the same.)

7. Find the locus of the points of bisection of equal chords in a circle.

8. Find the locus of the centres of circles which touch a given circle in a given point.

9. Find the locus of the middle point of a line drawn from a given point to meet a given circle.

10. Shew that the inscribed equilateral triangle is one-fourth of the circumscribed equilateral triangle.

11. A ladder slips down a wall: find the locus of its middle point.

12. If from two fixed points in the circumference of a circle two lines are drawn to intercept a given arc, the locus of their intersections is a circle.

13. Two chords of a circle which do not bisect each other do not both pass through the centre.

14. Two shillings are moved in the corner of a box so that each always touches one side, and they touch one another; find the locus of the point of contact.

15. Two circles cut one another, and lines are drawn through the points of section, and terminated by the circumferences; shew that the chords which join the extremities of these lines are parallel.

16. Two equal circles intersect in  $A$  and  $B$ , and any line  $BCD$  is drawn to cut both circles. Prove that

$$AC = AD.$$

17. Two equal circles intersect in  $A, B$ ; a third circle is drawn, with centre  $A$  and any radius less than  $AB$ , meeting the circles in  $C, D$ , on the same side of  $AB$ . Prove that  $B, C, D$  lie in one straight line.

18.  $ACD$ ,  $ADB$  are two segments of circles on the same base  $AB$ ; take any point  $C$  on the segment  $ACB$ , and join  $CA$ ,  $CB$ , and produce them if necessary to meet  $ADB$  in  $D$ ,  $E$ . Shew that the arc  $DE$  is constant.

19. If two circles cut each other, and from either point of intersection diameters be drawn, the extremities of these diameters and the other point of intersection shall be in the same straight line.

20. If a straight line that touches a circle be parallel to a chord of it, the point of contact will bisect the arc cut off by that chord.

21. Perpendiculars  $AD$ ,  $CE$  are let fall from the angles  $A$ ,  $C$  of the triangle  $ABC$  on the opposite sides. Prove that the angle  $ACE$  is equal to the angle  $ADE$ .

22. Two circles intersect in  $A$ ,  $B$ , and tangents  $AC$ ,  $AD$  are drawn to each circle, meeting circumferences in  $C$ ,  $D$ , prove that  $BC$ ,  $BD$  make equal angles with  $BA$ .

23. If one of two intersecting circles pass through the centre of the other, prove that the tangent to the first at the point of intersection, and the common chord, make equal angles with the radius to that point from the centre of the second.

24. Given base, altitude, and vertical angle, construct the triangle.

25. To draw a line from a given point such that the perpendicular on it from a given point shall have a given length.

26. In a given straight line to find a point at which a given straight line subtends a given angle.

27. Describe a circle to touch a given circle, and touch a given line in a given point.

28. Describe a circle of given radius to touch a given line, and have its centre on another given line.

29. A given chord of a circle is produced. Find a point in it from which the tangents to the circle shall have a given length.

30. With a given radius describe a circle touching two given circles.

31. Describe a triangle, having given the vertical angle and the segments of the base made by the line bisecting the vertical angle.

32. Given base, altitude, and radius of circumscribed circle, construct the triangle.

33. The triangle contained by the two tangents to a circle from any point and any other tangent that meets them and lies between the point and the circle has its perimeter double of either of the two tangents. Prove this ; and apply it to construct a triangle, having given the vertical angle, perimeter, and altitude.

34. Given the perimeter, the vertical angle, and the line bisecting the vertical angle, construct the triangle.

35. The chord  $AB$  is produced both ways equally to  $C$ ,  $D$ , and tangents  $CE$ ,  $DF$  are drawn on opposite sides of  $CD$ ; shew that  $EF$  bisects  $AB$ .

36. The three perpendiculars to the sides of a triangle drawn through their middle points meet in one point.

37. The three lines which join the angles of a triangle to the middle points of the opposite sides intersect in one point.

38. The three perpendiculars from the angles of a triangle on its opposite sides intersect in one point.

39. If two circles touch one another, the lines which join the extremities of parallel diameters towards opposite parts will intersect in the point of contact.

40. The circles described on the sides of a triangle as diameters intersect in the sides, or sides produced, of the triangle.

41. Equilateral triangles are described externally on the sides of a triangle; prove that the circles described about those triangles pass through one point.

42. The four common tangents to two circles which do not meet one another intersect, two and two, on the straight line which joins the centres of the circles.

43. Given the altitude, the bisector of the vertical angle, and the bisector of the base, to construct the triangle.

44. The three circles which pass through two angles of a triangle and the point of intersection of the perpendiculars of the triangle are each equal to the circumscribing circle.

45. If a triangle is equilateral, shew that the radii of the inscribed, the circumscribed, and an escribed circle are to one another as 1, 2, 3.

46. If circles are described with the vertices of a triangle as centres, and so as to pass through the points of contact of the inscribed circle with the adjacent sides, these three circles will touch one another.

47. Place a straight line of given length in a circle so that it shall be parallel to a given diameter of the circle.

48. Place (when possible) a straight line of given length in a circle so that it shall pass through a given point within or without the circle.

49. Given three points, describe circles from them as centres so that each may touch the other two.

50. On the side of any triangle equilateral triangles are described externally, and their vertices joined to the opposite vertices of the given triangle, shew that the joining lines pass through one point.

51.  $O$  is the centre of the circle inscribed in the triangle  $ABC$ , which touches  $AB, AC$  in  $C', B'$ ; if  $AO$  cuts the circle in  $P$ , and  $AO$  produced in  $P'$ , shew that  $P, P'$  are the centres of the inscribed and escribed circles of the triangle  $AB'C'$ .

52. Shew that the area of a triangle is equal to the rectangle contained by its semi-perimeter and the radius of the inscribed circle.

53. Of all the rectangles inscribable in a circle, shew that a square is the greatest.

54. Can a circle be inscribed in (1) a rectangle, (2) a parallelogram, (3) a rhombus?

55. Shew that the inscribed hexagon is three-fourths of the circumscribed hexagon.

56. To find four points such that the line joining every two may be perpendicular to the line joining the other two.

57. Shew that the six segments into which the points of contact of the escribed circles of a triangle divide the sides, may be arranged in three pairs of equal segments.

58. Inscribe an octagon in a given circle.
59. Describe a circle to intercept equal chords of any given length on three given straight lines.  
In how many ways may this problem be solved?
60. At any point in the circumference of the circle circumscribing a square, shew that one of the sides subtends an angle three times as great as the others.
61. Find the locus of points at which two sides of a square subtend equal angles.
62. Find the locus of points at which three sides of a square subtend equal angles.
63. If four straight lines intersect one another so as to form four triangles, prove that the four circumscribing circles will pass through one point.
64. Of all triangles that can be inscribed in a circle the greatest is the equilateral. Extend this to the case of a polygon of any number of sides.
65. Of all triangles that can be inscribed in a given triangle that whose angles are the feet of the perpendiculars of the original triangle has the smallest perimeter.
66. A straight line is divided into any two parts in  $C$ , and  $ADC$ ,  $CEB$  are equilateral triangles on the same side of  $AB$ . Find the locus of the intersection of  $AE$  and  $BD$ .
67. If from any point on the circle circumscribing a triangle perpendiculars be let fall upon the sides, the feet of these perpendiculars lie in one straight line.

# PROBLEM PAPERS ON THE TRIANGLE AND ITS ASSOCIATED CIRCLES.

## No. I.

1. The three bisectors of the angles of a triangle pass through one point ( $O$ ), and this is the centre of the inscribed circle.

2. The three straight lines which bisect one angle of a triangle and the other two exterior angles meet in one point ( $O'$ ), and this is the centre of an escribed circle of the triangle.

3. If the inscribed circle of the triangle  $ABC$  touches the sides opposite  $A, B, C$  in  $Q, R, S$ , and the circle escribed to the side opposite  $A$ , touches the sides or sides produced opposite to  $A, B, C$ , in  $X, Y, Z$ , prove that

$$AZ = AY = \frac{1}{2} \text{ perimeter of triangle,}$$

$$\text{and } QX = AB - AC,$$

$$\text{and } CR = BZ.$$

4. In the same figure prove that if  $a, b, c$  are the lengths of the sides,  $s$  half their sum,  $AR = s - a$ .

## No. II.

5. Prove that the three perpendiculars drawn to the sides of a triangle through their middle points meet in one point, which is the centre of the circumscribing circle.

6. Prove that the three perpendiculars drawn to the sides of a triangle from the opposite angles intersect in one point. {This point is often called the orthocentre.}

{ This may be deduced from (5), by drawing through each vertex a parallel to the opposite side. (Catalan) }



7. Prove that the three medians of a triangle intersect in one point (called the centre of gravity), which is a point of trisection of each median.

8. If  $G$  is the centre of gravity of the triangle  $ABC$ , prove that the triangles  $GAB$ ,  $GBC$ ,  $GCA$  are all equal.

### No. III.

9. If  $G$  is the centre of gravity of the triangle  $ABC$ , prove that

$$GA^2 + GB^2 + GC^2 = \frac{AB^2 + BC^2 + CA^2}{3}.$$

10. The centre of the circumscribing circle ( $I$ ), the centre of gravity ( $G$ ), and the point of intersection of the perpendiculars ( $P$ ), lie in one straight line, and

$$IG = \frac{1}{3}GP.$$

{If  $F$  is the middle point of  $AC$  and  $E$  of  $AB$ , prove the triangles  $EIF$ ,  $CPB$  similar, and  $IF = \frac{1}{2}BP$ .}

11. The circles which pass through two vertices of a triangle, and the intersection of the perpendiculars, will be equal to the circumscribing circle.

12. The angles  $BIC$ ,  $CIA$ ,  $AIC$  are respectively double of the angles at  $A$ ,  $B$ ,  $C$ .

### No. IV.

13. If  $Q$ ,  $R$ ,  $S$ , are the feet of the perpendiculars let fall from  $A$ ,  $B$ ,  $C$  on the opposite sides, prove that  $AQ$ ,  $BR$ ,  $CS$ , are the bisectors of the angles of the triangle  $QRS$ .

14. Prove that if  $I$  is the centre of the circumscribing circle, and  $Q, R, S$  as in (13),  $IA, IB, IC$  are respectively perpendicular to  $RS, SQ, QR$ .

15. From a centre  $O$  describe a circle; from a point  $G$  on its circumference describe a second circle cutting the former in  $B, C$ : and from a point  $I$  on the second circle describe a circle to touch  $BC$ . Prove that the other tangents from  $B, C$  to the third circle will intersect on the circumference of the first.

16. Hence shew that if  $I, I'$  are centres of the inscribed and escribed circles of a triangle,  $II'$  is bisected by the circumference of the circumscribing circle.

#### No. V.

17.  $I$  is the centre of the circumscribing circle,  $P$  the intersection of the perpendiculars;  $E, F, G$  the middle points of  $BC, CA, AB$ ;  $Q, R, S$  the feet of the perpendiculars from  $A, B, C$  on those sides:  $H$  the middle point of  $IP$ . Prove that  $H$  is the centre of a circle which bisects  $PA, PB, PC$ , and that its radius is half that of the circumscribing circle.

18. If  $L, M, N$  are the middle points of  $PA, PB, PC$ , prove that  $IE = PL$ , and hence that  $EL$  is bisected in  $H$ .

19. Hence prove that the circle  $LMN$  also passes through  $E, F, G$ , and through  $Q, R, S$ .

[This circle is therefore called the nine point circle.]

NOTE. The advanced student will do well to get Catalan's *Théorèmes et Problèmes de Géométrie Élémentaire*.

## BOOK IV.

### FUNDAMENTAL PROPOSITIONS OF PROPORTION.

#### SECTION I.

##### OF RATIO AND PROPORTION.

[Although a complete treatment of Proportion, such as that contained in this Book, is indispensable to a sound knowledge of Geometry, Book V. may be read immediately after Book III. by students who are acquainted with the treatment of Ratio and Proportion given in books on Arithmetic and Algebra.]

##### [Notation.]

In what follows, large Roman letters, A, B, etc., are used to denote magnitudes, and where the pairs of magnitudes compared are both of the same kind they are denoted by letters taken from the early part of the alphabet, as A, B compared with C, D; but where they are or may be of different kinds, from different parts of the alphabet, as A, B compared with P, Q or X, Y. Small Italic letters  $m, n$ , etc., denote whole numbers. By  $m \cdot A$  or  $mA$  is denoted the  $m$ th multiple of A, and it may be read as  $m$  times A. The product of the numbers  $m$  and  $n$  is denoted by  $mn$ , and it is assumed that  $mn = nm$ . The combination  $m \cdot nA$  denotes the  $m$ th multiple of the  $n$ th multiple of A and may be read as  $m$  times  $nA$ , and  $mnA$  or  $m \cdot n \cdot A$  as  $mn$  times A. By  $(m + n)A$  is denoted  $m + n$  times A.]

*Def. 1.* One magnitude is said to be a *multiple* of another magnitude when the former contains the latter an exact number of times.

According as the number of times is 1, 2, 3... $m$ , so is the multiple said to be the 1st, 2nd, 3rd,... $m$ th.

*Def. 2.* One magnitude is said to be a *measure* or *part* of another magnitude when the former is contained an exact number of times in the latter.

The following property of multiples is axiomatic:—

1. As  $A > = \text{or} < B$ , so is  $mA > = \text{or} < mB$  (*Euc. Ax. 1 & 3*).

The converse necessarily follows, so that

2. As  $mA > = \text{or} < mB$ , so is  $A > = \text{or} < B$  (*Euc. Ax. 2 & 4*).

The following theorems are easily proved:—

3.  $mA + mB + \dots = m(A + B + \dots)$  (*Euc. v. 1*).

4.  $mA - mB = m(A - B)$  ( $A$  being greater than  $B$ ) (*Euc. v. 5*).

5.  $mA + nA = (m + n)A$  (*Euc. v. 2*).

6.  $mA - nA = (m - n)A$  ( $m$  being greater than  $n$ ) (*Euc. v. 6*).

7.  $m \cdot nA = mn \cdot A = nm \cdot A = n \cdot mA$  (*Euc. v. 3*).

*Def. 3.* The *ratio* of one magnitude to another of the same kind is the relation of the former to the latter in respect of *quantuplicity*.

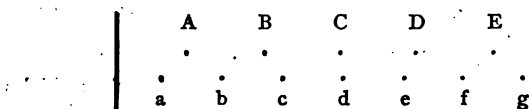
The ratio of  $A$  to  $B$  is denoted thus,  $A : B$ , and  $A$  is called the *antecedent* of the ratio,  $B$  the *consequent*.

The *quantuplicity* of  $A$  with respect to  $B$  may be estimated by examining how the multiples of  $A$  are distributed among the multiples of  $B$ , when both are arranged in ascending order of magnitude and the series of multiples continued without limit.

Obs. This interdistribution of multiples is *definite* for two given magnitudes  $A$  and  $B$ , and is different from that for  $A$  and  $C$ , if  $C$  differ from  $B$  by any magnitude however small. See Th. 4.

This is a very important definition, and may be illustrated as follows. (The illustration is due to De Morgan.)

In front of a row of pillars in a street is a row of palings, the



pillars being  $A, B$ , &c., the palings  $a, b, c, \dots$  in the figure.

Then there is a certain interdistribution of the pillars among the palings, or in other words of the multiples of the distance between the pillars among the multiples of the distance between the palings. The 2nd pillar lies between the 2nd and 3rd palings, the 5th pillar between the 6th and 7th palings, and so on.

Now if the distance between the palings or between the pillars were altered by any quantity however small, then the distribution would be changed if the series were continued without limit. For if the distance between the palings were changed by a distance equal to say the  $n$ th part of the distance between the pillars, then the  $n$ th paling would be changed by the whole distance between two pillars, and therefore its position among the pillars would be changed.

*Def. 4.* The ratio of two magnitudes is said to be equal to that of two other magnitudes (whether of the same or of a different kind from the former), when any equimultiples whatever of the antecedents of the ratios being taken and likewise any equimultiples whatever of the consequents, the multiple of one antecedent is greater than, equal to, or less than that of its consequent, according as that of the other antecedent is greater than, equal to, or less than that of its consequent.

Or in other words:

The ratio of  $A$  to  $B$  is equal to that of  $P$  to  $Q$ , when  $mA$  is greater than, equal to, or less than  $nB$ , according as  $mP$  is greater than, equal to, or less than  $nQ$ , whatever whole numbers  $m$  and  $n$  may be.

It is an immediate consequence that:

The ratio of  $A$  to  $B$  is equal to that of  $P$  to  $Q$ ; when,  $m$  being any number whatever, and  $n$  another number determined so that either  $mA$  is between  $nB$  and  $(n+1)B$  or equal to  $nB$ , according as  $mA$  is between  $nB$  and  $(n+1)B$  or is equal to  $nB$ , so is  $mP$  between  $nQ$  and  $(n+1)Q$  or equal to  $nQ$ .

The definition may also be expressed thus:

The ratio of  $A$  to  $B$  is equal to that of  $P$  to  $Q$  when the multiples of  $A$  are distributed among those of  $B$  in the same manner as the multiples of  $P$  are among those of  $Q$ .

That is, if a model were constructed of the pillars and palings, it would be correct, or the ratio of the distances of pillars and palings in the street is the same as the ratio of the distance of pillars and palings in the model, if every pillar in the model fell between the same palings in the model, as the corresponding pillar in the street did among the palings in the street, the street being supposed to be of indefinite length.

It will be observed that this is a method of ascertaining whether four magnitudes are in proportion which is wholly independent of any arithmetical representation of the numbers.

*Def. 5.* The ratio of two magnitudes is greater than that of two other magnitudes, when equimultiples of the antecedents and equimultiples of the consequents can be found such that, while the multiple of the antecedent of the first is greater than or equal to that of its consequent, the multiple of the antecedent of the other is not greater or is less than that of its consequent.

Or in other words :

The ratio of A to B is greater than that of P to Q, when whole numbers  $m$  and  $n$  can be found, such that, while  $mA$  is greater than  $nB$ ,  $mP$  is not greater than  $nQ$ , or while  $mA = nB$ ,  $mP$  is less than  $nQ$ .

*Def. 6.* When the ratio of A to B is equal to that of P to Q, the four magnitudes are said to be *proportionals* or to form a *proportion*. The proportion is denoted thus:

$$A : B :: P : Q,$$

which is read, "A is to B as P is to Q." A and Q are called the *extremes*, B and P the *means*, and Q is said to be the *fourth proportional* to A, B and P.

The antecedents A, P are said to be *homologous*\*, and so are the consequents, B, Q.

\* That is, occupy the same position in the ratio.

*Def. 7.* Three magnitudes (A, B, C) of the same kind are said to be proportionals, when the ratio of the first to the second is equal to that of the second to the third: that is when  $A : B :: B : C$ .

In this case C is said to be the *third proportional* to A and B, and B the *mean proportional* between A and C.

*Def. 8.* The ratio of any magnitude to an equal magnitude is said to be a *ratio of equality*. If A be greater than B, the ratio  $A : B$  is said to be a *ratio of greater inequality*, and the ratio  $B : A$  a *ratio of less inequality*. Also the ratios  $A : B$  and  $B : A$  are said to be *reciprocal* to one another.

#### THEOREM I.

*Ratios that are equal to the same ratio are equal to one another.*

*Proof.* Let  $A : B :: P : Q$ , and also  $A : B :: X : Y$ , then shall  $P : Q :: X : Y$ .

For since  $mA > < nB$  according as  $mP > < nQ$   
(Def. 4.)

and  $mA > < nB$  according as  $mX > < nY$ ,

therefore  $mP > < nQ$  according as  $mX > < nY$ ,

and therefore (Def. 4)  $P : Q :: X : Y$ .

#### THEOREM 2.

*If two ratios are equal, as the antecedent of the first is greater than, equal to, or less than its consequent, so is the antecedent of the second greater than, equal to, or less than its consequent.*

*Proof.* Let  $A : B :: P : Q$ , then as  $A \geq B$  so is  $P \geq Q$ .

For by Def. 4, as  $mA \geq nB$  so  $mP \geq nQ$ , whatever integers  $m$  and  $n$  are. Let  $m$  and  $n$  each equal 1; then as  $A \geq B$  so  $P \geq Q$ .

THEOREM 3.

*If two ratios are equal, their reciprocal ratios are equal.*

*Proof.* Let  $A : B :: P : Q$ , then  $B : A :: Q : P$ .

For, since the multiples of  $A$  are distributed among those of  $B$  as the multiples of  $P$  among those of  $Q$ , the multiples of  $B$  are distributed among those of  $A$  as the multiples of  $Q$  among those of  $P$ ; and therefore

$$B : A :: Q : P. \quad (\text{Def. 4.})$$

THEOREM 4.

*If the ratios of each of two magnitudes to a third magnitude be taken, the first ratio will be greater than, equal to, or less than the other as the first magnitude is greater than, equal to, or less than the other: and if the ratios of one magnitude to each of two others be taken, the first ratio will be greater than, equal to, or less than the other as the first of the two magnitudes is less than, equal to, or greater than the other.*

*Proof.* Let  $A, B, C$  be three magnitudes of the same kind, then

$$A : C \geq B : C, \text{ as } A \geq B,$$

$$\text{and } C : A \geq C : B, \text{ as } A \leq B.$$

If  $A = B$ , it follows directly from Def. 4 that  $A : C :: B : C$  and  $C : A :: C : B$ .

If  $A > B$ ,  $m$  can be found such that  $mB$  is less than  $mA$  by a greater magnitude than  $C$ .



Hence if  $mA$  be between  $nC$  and  $(m+1)C$ , or if  $mA=nC$ ,  $mB$  will be less than  $nC$ , whence (Def. 6)  $A : C > B : C$ ;

Also, since  $nC > mB$  while  $nC$  is not  $> mA$  (Def. 6)  $C : B > C : A$  or  $C : A < C : B$ .

If  $A < B$ , then  $B > A$  and therefore  $B : C > A : C$ , that is  $A : C < B : C$ , and so also  $C : A > C : B$ .

*COR.* The converses of both parts of the proposition are true, since the "Rule of Conversion" is applicable.

#### THEOREM 5.

*The ratio of equimultiples of two magnitudes is equal to that of the magnitudes themselves.*

*Proof.* Let  $A, B$  be two magnitudes, then  $mA : mB :: A : B$ .

For as  $pA > \text{or} < qB$ , so is  $m \cdot pA > \text{or} < m \cdot qB$ ; but  $m \cdot pA = p \cdot mA$  and  $m \cdot qB = q \cdot mB$ , therefore as  $pA > \text{or} < qB$ , so is  $p \cdot mA > \text{or} < q \cdot mB$ , whatever be the values of  $p$  and  $q$ , and hence  $mA : mB :: A : B$ .

#### THEOREM 6.

*If two magnitudes ( $A, B$ ) have the same ratio as two whole numbers ( $m, n$ ), then  $nA = mB$ : and conversely if  $nA = mB$ ,  $A$  has to  $B$  the same ratio as  $m$  to  $n$ .*

*Proof.* Of  $A$  and  $m$  take the equimultiples  $nA$  and  $n \cdot m$ , and of  $B$  and  $n$  take the equimultiples  $mB$  and  $m \cdot n$ , then since

$$A : B :: m : n,$$

therefore as  $nA$  is  $> < mB$ , so is  $nm > < m \cdot n$ ;

but since  $n \cdot m = m \cdot n$ , it follows (Def. 4) that  $nA = mB$ .

Again since  $mB : nB :: m : n$  we have, if  $nA = mB$ ,  $nA : nB :: m : n$ ; whence it follows (Theor. 5) that  $A : B :: m : n$ .

COR. If  $A : B :: P : Q$  and  $nA = mB$ , then  $nP = mQ$ ; whence if  $A$  be a multiple, part, or multiple of a part of  $B$ ,  $P$  is the same multiple, part, or multiple of a part of  $Q$ .

## THEOREM 7.

*If four magnitudes of the same kind be proportionals, the first will be greater than, equal to, or less than the third, according as the second is greater than, equal to, or less than the fourth.*

*Proof.* Let  $A : B :: C : D$ .

Then if  $A = C$ ,  $A : B :: C : B$ , and therefore  $C : D :: C : B$ , whence  $B = D$ .

Also if  $A > C$ ,  $A : B > C : B$ , and therefore  $C : D > C : B$ , whence  $B > D$ .

Again if  $A < C$ ,  $A : B < C : B$ , and therefore  $C : D < C : B$ , whence  $B < D$ .

## THEOREM 8.

*If four magnitudes of the same kind be proportionals, the first will have to the third the same ratio as the second to the fourth.*

*Proof.* Let  $A : B :: C : D$ , then  $A : C :: B : D$ .

For (Th. 6)  $mA : mB :: A : B$  and  $nC : nD :: C : D$ ;  
therefore  $mA : mB :: nC : nD$ ,  
whence (Th. 7)  $mA \geq$  or  $< nC$ , as  $mB \geq$  or  $< nD$ ,  
and this being true for all values of  $m$  and  $n$ ,

$$A : C :: B : D.$$

## THEOREM 9.

*If any number of magnitudes of the same kind be proportionals, as one of the antecedents is to its consequent, so shall the sum of the antecedents be to the sum of the consequents.*

*Proof.* Let  $A : B :: C : D :: E : F$ , then  $A : B :: A + C + E : B + D + F$ .

For as  $mA \geq nB$ , so is  $mC \geq nD$ ,  
and so also is  $mE \geq nF$ ; whence it follows  
that so also is  $mA + mC + mE \geq nB + nD + nF$ ,  
and therefore so is  $m(A + C + E) \geq n(B + D + F)$ ,  
whence  $A : B :: A + C + E : B + D + F$ .

## THEOREM 10.

*If two ratios are equal, the sum or difference of the antecedent and consequent of the first has to the consequent the same ratio as the sum or difference of the antecedent and consequent of the other has to its consequent.*

*Proof.* Let  $A : B :: P : Q$ , then  $A + B : B :: P + Q : Q$   
and  $A - B : B :: P - Q : Q$ .

For,  $m$  being any whole number,  $n$  may be found such that either  $mA$  is between  $nB$  and  $(n + 1)B$  or  $mA = nB$ ,  
and therefore  $mA + mB$  is between  $mB + nB$  and  $mB + (n + 1)B$  or  $= mB + nB$ ;  
but  $mA + mB = m(A + B)$  and  $mB + nB = (m + n)B$ ,  
therefore  $m(A + B)$  is between  $(m + n)B$  and  $(m + n + 1)B$   
or  $= (m + n)B$ .

But as  $mA$  is between  $nB$  and  $(n + 1)B$  or  $= nB$ ,  
so is  $mP$  between  $nQ$  and  $(n + 1)Q$  or  $= nQ$ ;  
whence as  $m(A + B)$  is between  $(m + n)B$  and  $(m + n + 1)B$   
or  $= (m + n)B$ ,

so is  $m(P + Q)$  between  $(m + n)Q$  and  $(m + n + 1)Q$  or  $= (m + n)Q$ ,

and therefore, since  $m$  is any whole number whatever,

$$A + B : B :: P + Q : Q.$$

By like reasoning subtracting  $mB$  from  $mA$  and  $nB$  when  $A > B$  and therefore  $m < n$ , and subtracting  $mA$  and  $nB$  from  $mB$  when  $A < B$  and therefore  $m > n$ , it may be proved that

$$A - B : B :: P - Q : Q.$$

**COR.** *If two ratios are equal, the sum or difference of the antecedent and consequent of the first has to their difference or sum the same ratio as the sum or difference of the antecedent and consequent of the second has to their difference or sum.*

#### THEOREM II.

*If two ratios are equal, and equimultiples of the antecedents and also of the consequents are taken, the multiple of the first antecedent has to that of its consequent the same ratio as the multiple of the other antecedent has to that of its consequent.*

*Proof.* Let  $A : B :: P : Q$ , then  $mA : nB :: mP : nQ$ .

For  $pm \cdot A > \text{or} < qn \cdot B$ , as  $pm \cdot P > \text{or} < qn \cdot Q$ , and therefore  $p \cdot mA > \text{or} < q \cdot nB$ , as  $p \cdot mP > \text{or} < q \cdot nQ$ , whence,  $p, q$  being any numbers whatever,

$$mA : nB :: mP : nQ.$$

#### THEOREM 12.

*If there be two sets of magnitudes, such that the first is to the second of the first set as the first to the second of the other set, and the second to the third of the first set as the second to the third of the other, and so on to the last magnitude: then the first is to the last of the first set as the first to the last of the other.*

*Proof.* Let the two sets of three magnitudes be  $A, B, C$  and  $P, Q, R$ ,

and let  $A : B :: P : Q$  and  $B : C :: Q : R$ ,

then  $A : C :: P : R$ .

[Lemma.—As  $A > =$  or  $< C$ , so is  $P < =$  or  $< R$ .

For if  $A > C$ ,  $A : B > C : B$  and  $C : B :: R : Q$ ,

therefore  $P : Q > R : Q$ , whence  $P > R$ .

Similarly if  $A = C$  or if  $A < C$ . Hence the lemma is proved.]

By Theor. 6,  $mA : mB :: mP : mQ$ , and by Theor. 11,  $mB : nC :: mQ : nR$ , whence by the lemma as  $mA > =$  or  $< nC$ , so is  $mP > =$  or  $< nR$ , and therefore,  $m$  and  $n$  being any numbers whatever,

$$A : C :: P : R.$$

If there be more magnitudes than three in each set, as  $A, B, C, D$  and  $P, Q, R, S$ ;

then, since  $A : B :: P : Q$  and  $B : C :: Q : R$ ,

therefore  $A : C :: P : R$ ; but  $C : D :: R : S$ ,

and therefore  $A : D :: P : S$ .

Q. E. D.

COR. If  $A : B :: Q : R$  and  $B : C :: P : Q$ , then  $A : C :: P : R$ .

*Proof.* Let  $S$  be a fourth proportional to  $Q, R, P$ ,

then  $Q : R :: P : S$ ,

therefore  $Q : P :: R : S$ , (Th. 8.)

and  $P : Q :: S : R$ . (Th. 3.)

Hence  $A : B :: P : S$  and  $B : C :: S : R$ ,

therefore  $A : C :: P : R$ .

*Def. 9.* If there are any number of magnitudes of the same kind, the first is said to have to the last the ratio *compounded* of the ratios of the first to the second, of the second to the third, and so on to the last magnitude.

*Def. 10.* If there are any number of ratios, and a set of magnitudes is taken such that the ratio of the first to the second is equal to the first ratio, and the ratio of the second to the third is equal to the second ratio, and so on, then the first of the set is said to have to the last the ratio *compounded* of the original ratios.

*Obs.* From these definitions it follows, by Theor. 12, that if there be two sets of ratios equal to one another, each to each, the ratio compounded of the ratios of the first set is equal to that compounded of the ratios of the other set.

Also that the ratio compounded of a given ratio and its reciprocal is the ratio of equality.

*Def.* When two ratios are equal, the ratio compounded of them is called the *duplicate* ratio of either of the original ratios.

*Def.* When three ratios are equal, the ratio compounded of them is called the *triplicate* ratio of any one of the original ratios.

## SECTION II.

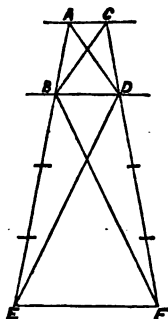
### FUNDAMENTAL GEOMETRICAL PROPOSITIONS.

#### LEMMA.

If on two straight lines  $AB$ ,  $CD$  cut by two parallel straight lines  $AC$ ,  $BD$  equimultiples of the intercepts respectively are taken; then the line joining the points of division will be parallel to  $AC$  or  $BD$ .

Let  $BE$ ,  $DF$  be equimultiples of  $AB$ ,  $CD$ ;

Then will  $EF$  be parallel to  $BD$ .



*Proof.* Join  $AD$ ,  $DE$ ,  $BC$ ,  $BF$ .

Since the triangles  $ABD$ ,  $CBD$  are on the same base  $BD$ , and of the same altitude, they are equal.

(II. 2. Cor. 1.)

Also whatever multiple  $BE$  is of  $AB$ , the same multiple is the triangle  $DBE$  of the triangle  $ABD$ , and the triangle  $DBF$  of the triangle  $CBD$ :

Therefore the triangle  $EBD$  = the triangle  $FBD$ , and they are on the same base  $BD$ ;

therefore  $EF$  is parallel to  $BD$ .

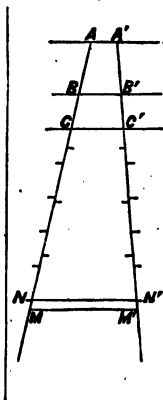
(II. 2 Cor. 3.)

#### THEOREM I.

*If two straight lines are cut by three parallel straight lines, the intercepts on the one are to one another in the same ratio as the corresponding intercepts on the other,*

*Proof.* Let the three parallel lines  $AA'$ ,  $BB'$ ,  $CC'$ , cut other two lines in  $A$ ,  $B$ ,  $C$ , and  $A'$ ,  $B'$ ,  $C'$  respectively ;

then  $AB : BC :: A'B' : B'C'$ .



On the line  $ABC$  take  $BM = m \cdot AB$  and  $BN = n \cdot BC$ ,  $M$  and  $N$  being taken on the same side of  $B$ . Also on the line  $A'B'C'$  take  $B'M' = m \cdot A'B'$  and  $B'N' = n \cdot B'C'$ ,  $M'$ ,  $N'$  being on the same side of  $B'$  as  $M$ ,  $N$  are of  $B$ . Then by the Lemma  $MM'$  and  $NN'$  are both parallel to  $BB'$ . Hence, whatever be the values of  $m$  and  $n$ ,

as  $BM$  (or  $m \cdot AB$ ) is greater than, equal to, or less than  $BN$  (or  $n \cdot BC$ ),

so is  $B'M'$  (or  $m \cdot A'B'$ ) greater than, equal to, or less than  $B'N'$  (or  $n \cdot B'C'$ ),

therefore  $AB : BC :: A'B' : B'C'$ .

It will be observed that the reasoning holds good, whether  $B$  be between  $A$  and  $C$  or beyond  $A$  or beyond  $C$ .



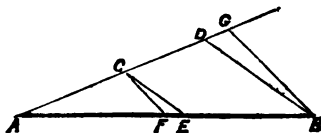
**COR. 1.** *If the sides of a triangle are cut by a straight line parallel to the base, the segments of one side are to one another in the same ratio as the segments of the other side.*

**COR. 2.** *If two straight lines are cut by four parallel straight lines the intercepts on the one are to one another in the same ratio as the corresponding intercepts on the other.*

### THEOREM 2.

*A given finite straight line can be divided internally into segments having any given ratio, and also externally into segments having any given ratio except the ratio of equality: and in each case there is only one such point of division.*

*Proof.* Let  $AB$  be the given straight line and, since any given ratio may be expressed as the ratio of two straight



lines, let  $AC$ ,  $CD$  be two lines having the given ratio taken on an indefinite line drawn from  $A$  making any angle with  $AB$ ; join  $DB$ ; draw  $CE$  parallel to  $DB$ ; then will  $CE$  (Theor. 1) divide  $AB$  internally in  $E$  in the given ratio. If it could be divided internally at  $F$  in the same ratio,  $BG$  being drawn parallel to  $CF$  to meet  $AD$  in  $G$ ,  $AF$  would be to  $FB$  as  $AC$  to  $CG$ , and therefore not as  $AC$  to  $CD$ . Hence  $E$  is the only point which divides  $AB$  internally in the given ratio. If  $CD$  be taken so that  $A$  and  $D$  are on the same

side of C, the like construction will determine the external point of division. In this case the construction will fail, if  $CD = AC$ . A like demonstration will shew that there can be only one point of external division in the given ratio.

## THEOREM 3.

*A straight line which divides the sides of a triangle proportionally is parallel to the base of the triangle.*

Let  $AP : PB :: AQ : QC$ ;  
then will  $PQ$  be parallel to  $BC$ .

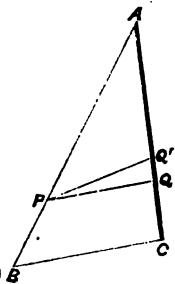
For if not, if possible let some other line  $PQ'$  be parallel to  $BC$ .

Then  $AP : PB :: AQ' : Q'C$ ;  
(Th. I, Cor. I.)

but  $AP : PB :: AQ : QC$ ; (Hyp.)

Therefore  $AQ' : Q'C :: AQ : QC$ ;  
which is impossible ;

and therefore  $PQ$  is parallel to  $BC$ .



## THEOREM 4.

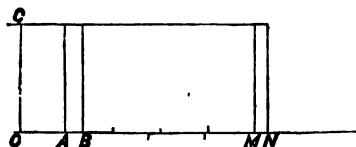
*Rectangles of equal altitude are to one another in the same ratio as their bases.*

*Proof.* Let AC, BC be two rectangles having the common side OC and their bases OA, OB on the same side of OC.

In the line OAB indefinitely produced, take  $OM = m \cdot OA$  and  $ON = n \cdot OB$ , and complete the rectangles MC and NC.

W.

Then  $MC = m \cdot AC$  and  $NC = n \cdot BC$ :

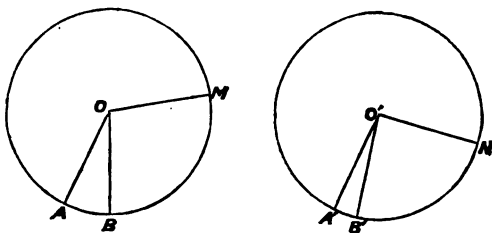


and it is plain that as  $OM$  is greater than, equal to, or less than  $ON$ , so is  $MC$  greater than, equal to, or less than  $AC$ ; whence the rectangle  $AC$  : the rectangle  $BC ::$  base  $OA$  : base  $OB$ .

*COR. Parallelograms or triangles of the same altitude are to one another as their bases.*

#### THEOREM 5.

*In the same circle or in equal circles angles at the centre and sectors are to one another as the arcs on which they stand.*



*Proof.* Let  $O, O'$  be the centres of two equal circles, and let  $AB, A'B'$  be any two arcs in them; then shall the angle or sector  $AOB$  be to the angle or sector  $A'O'B'$  as the arc  $AB$  is to the arc  $A'B'$ .

For let  $AM$  be an arc  $= m \cdot AB$ , then the angle or sector between  $OA$  and  $OM$  (reckoned correspondingly to the arc  $AM$ ) will be  $m$  times the angle or sector  $AOB$ .

And let  $A'N$  be an arc  $= n \cdot A'B'$ , and  $A'O'N$  an angle or sector  $n$  times the angle or sector  $A'O'B'$  :

and according as  $AM$  is  $>$ ,  $=$ , or  $<$   $A'N$ ,

so is the angle or sector  $AO'M$   $>$ ,  $=$ , or  $<$  the angle or sector  $A'O'N$ ;

and therefore as  $AB : A'B' ::$  angle or sector  $AOB$  : angle or sector  $A'O'B'$ .

## BOOK V.

### PROPORTION.

#### INTRODUCTION.

[For the use of those for whom it may be thought well to defer the study of the complete, but more difficult, mode of treatment of Proportion in Book IV., the following Definitions and Propositions referred to in that Book are here collected, with an indication of the principles of an incomplete mode of treatment by which they may be established for commensurable magnitudes.]

*Def. 1.* One magnitude is said to be a *multiple* of another magnitude when the former contains the latter an exact number of times. According as the number of times is 1, 2, 3... $m$ , so is the multiple said to be the 1st, 2nd, 3rd... $m$ th.

*Def. 2.* One magnitude is said to be a *measure* or *part* of another magnitude when the former is contained an exact number of times in the latter.

*Def. 3.* If a magnitude can be found which is a measure of two or more magnitudes, these magnitudes are said to be *commensurable*, and the first magnitude is said to be a *common measure* of the others.

It is easy to prove that commensurable magnitudes have also a common multiple, and conversely that magnitudes which have a common multiple are commensurable.

A *measure* of a line is any line which is contained in it an exact number of times. Thus an inch is a measure of a foot ; and a yard is a measure of a mile. So too the measure of an area is any area which is contained an exact number of times in it. A square inch is thus a measure of a square yard. *A measure is therefore an aliquot part of any magnitude which it measures.* The length of a line, the extent of an area, or any other magnitude, is completely known when we know a measure of it, and how many times it contains that measure.

In measuring any magnitude we take some standard to measure by. Thus in measuring length we take a yard, or a foot, or an inch. In measuring solids we take a cubic inch, a cubic foot, or the like. The standard so taken is called the *unit*. It may be a precise measure of the magnitude measured, or it may not. The number, whether whole or fractional, which expresses how many times a magnitude contains a certain unit is called the *numerical value* of that magnitude in terms of that unit. Thus in speaking of a line as 7 yards long, a yard is the unit of length, and the numerical value of the line in terms of that unit is 7.

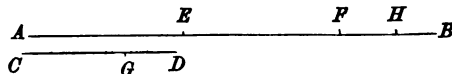
Two lines or magnitudes of the same kind are said to have a *common measure* when there exists a unit of which they can both be expressed as multiples. Thus 15 inches and 1 foot have a common measure, for with the unit 3 inches, their numerical values would be 5 and 4; and with the unit 1 inch their numerical values would be 15 and 12. All whole numbers have unity as a common measure.

The following problem gives a method of finding the greatest common measure of two magnitudes, if any common measure exists, and illustrates the familiar Arithmetical method.

#### PROBLEM.

*To find the greatest common measure of two magnitudes, if they have a common measure.*

Let  $AB$  and  $CD$  be the two magnitudes. From  $AB$  the greater cut



off parts,  $AE, EF, \dots$  each equal to  $CD$  the less, leaving a remainder  $FB$  which is less than  $CD$ .

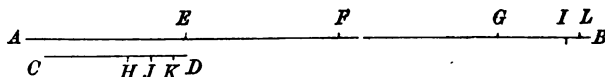
From  $CD$  cut off parts,  $CG...$ , equal to  $FB$ , leaving a remainder  $GD$  less than  $FB$ .

From  $FB$  cut off parts  $FH, HB...$  equal to  $GD$ : and continue this process until a remainder  $GD$  is found which is contained *an exact number* of times in the previous remainder, so that no further remainder is left. The last remainder is then the greatest common measure.

For, firstly, since  $GD$  measures  $FB$ , it also measures  $CG$ ; and therefore measures  $CD$ . But  $CD = AE$  and  $EF$ ; and therefore  $GD$  measures  $AE, EF$  and  $FB$ ; that is it measures  $AB$ . Hence  $GD$  is a common measure of  $AB$  and  $CD$ .

And again, since every measure of  $CD$  and  $AB$  must measure  $AF$ , it must measure  $FB$  or  $CG$ , and therefore also  $GD$ : hence the common measure cannot be greater than  $GD$ ; that is  $GD$  is the *greatest* common measure.

So also, in the figure adjoining, the first remainder is  $GB$ ; the



second  $HD$ ; the third  $IB$ ; the fourth  $KD$ , which is contained exactly twice in  $IB$ . Hence  $KD$  is the greatest common measure, and it will be seen to be contained twice in  $IB$ , and therefore five times in  $HD$ , seven times in  $GB$ , 12 times in  $CD$ , and 43 times in  $AB$ .

Hence  $AB$  and  $CD$  have as their numerical values 43 and 12 in terms of the unit  $KD$ .

COR. Every measure of  $KD$  is a common measure of  $AB$  and  $CD$ .

When magnitudes have a common measure they are called *commensurable*. But it is very frequently the case in Geometrical figures, that lines and other magnitudes have no common measure; the process above given continuing indefinitely; the remainder becoming smaller at each step of the process but never actually disappearing. In this case the lines are said to be *incommensurable*.

*Def. 4.* The *ratio* of one magnitude to another of the same kind is the relation of the former to the latter in respect of *quantuplicity*.

The ratio of A to B is denoted thus,  $A : B$ , and A is called the *antecedent*, B the *consequent*.

The complete examination of the nature of the comparison of two magnitudes according to quantuplicity is contained in Book IV. For numbers, and for magnitudes generally, *so far as they are commensurable* (and it is to be noted that this is not the *normal*, but the *exceptional*, case), the comparison may be made in a more simple manner either

(1) (As is usual in Arithmetic) by considering what multiple, part, or multiple of a part one magnitude is of the other ;  
or (2) by considering what multiples of the two magnitudes are equal to one another.]

*Def. 5.* When the ratio  $A : B$  is equal to the ratio  $P : Q$ , *i.e.* either

(1) When A is the same multiple, part, or multiple of a part of B as P is of Q ; or,

(2) When like multiples of A and P are equal respectively to like multiples of B and Q ;

the four magnitudes are said to be *proportionals*, or to form *proportion*.

The equality of the ratios is denoted by the symbol  $::$  ; and the proportion thus,  $A : B :: P : Q$ , which is read A is to B as P is to Q.

A and Q are called the *extremes*, B and P the *means*, and Q is said to be the *fourth proportional* to A, B and P. The antecedents A, P are said to be *homologous* to one another, and so also are the consequents.



*Def. 6.* If  $A, B, C$  are three magnitudes of the same kind such that  $A : B :: B : C$ ,  $B$  is said to be the *mean proportional* between  $A$  and  $C$ , and  $C$  the *third proportional* to  $A$  and  $B$ .

*Def. 7.* If there are two ratios  $A : B, P : Q$ , and  $C$  be taken such that  $B : C :: P : Q$ , then  $A$  is said to have to  $C$  a ratio *compounded* of the ratios  $A : B, P : Q$ . Thus if there are three magnitudes  $A, B, C$ , then  $A$  has to  $C$  the ratio compounded of the ratios  $A : B, B : C$ .

*Def. 8.* A ratio compounded of two equal ratios is called the *duplicate* of either of these ratios.

It is evident that different ratios cannot have the same duplicate ratio.

#### GENERAL PROPOSITIONS ON PROPORTION.

[All these propositions admit of obvious algebraical proof.]

(1.) Ratios that are equal to the same ratio are equal to one another.

(2.) Equal magnitudes have the same ratio to the same or to equal magnitudes.

(3.) Magnitudes that have the same ratio to the same or equal magnitudes are equal.

(4.) The ratio of two magnitudes is equal to that of their halves or doubles.

(5.) If  $A : B :: P : Q$ , then  $B : A :: Q : P$ .

(invertendo)

(6.) If  $A : B :: C : D$ , all the four being of the same kind,  
then  $A : C :: B : D$ .

(alternando)

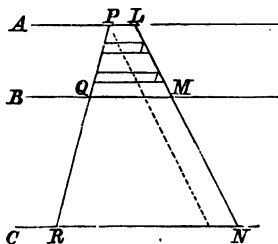
(7.) If  $A : B :: P : Q$ ,  
 then  $A + B : B :: P + Q : Q$ , (componendo)  
 and  $A - B : B :: P - Q : Q$ . (dividendo)

(8.) If  $A : B :: C : D :: E : F$ ,  
 then  $A + C + E : B + D + F :: A : B$ . (addendo)

(9.) If  $A : B :: P : Q$   
 and  $B : C :: Q : R$ ,  
 then  $A : C :: P : R$ . (ex æquali)

## THEOREM I.

*If two straight lines are cut by three parallel straight lines, the intercepts on the one are to one another in the same ratio as the corresponding intercepts on the other.*



Let  $A, B, C$  be the three parallels,  $PQR, LMN$  any two lines intersected by them; then shall

$$PQ : QR :: LM : MN.$$

*Proof.* Let  $PQ, QR$  be commensurable, and contain their common measure  $m$  and  $n$  times respectively: and through the points of division let lines parallel to  $A$  be drawn to meet  $LM$ .

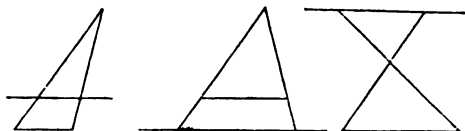
Then (by 1. 32)  $LM$  and  $MN$  will be divided into  $m$  and  $n$  equal parts respectively,

therefore

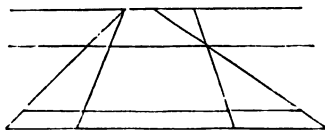
$$PQ : QR :: m : n$$

$$:: LM : MN.$$

**COR. 1.** *If the sides of a triangle are cut by a straight line parallel to the base, the segments of one side are to one another in the same ratio as the segments of the other side.*



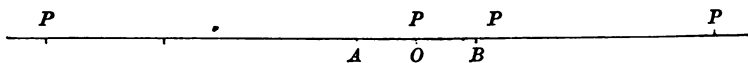
**COR. 2.** *If two straight lines are cut by four parallel straight lines the intercepts on the one are to one another in the same ratio as the corresponding intercepts on the other.*



### THEOREM 2.

*A given finite straight line can be divided internally into segments having any given ratio, and also externally into segments having any given ratio except the ratio of equality: and in each case there is only one such point of division.*

Let  $AB$  be the given finite straight line, and let  $O$  be the point of bisection of  $AB$ ; then if  $P$  is at  $O$  the ratio  $\frac{PA}{PB} = 1$ .



Conceive the point  $P$  to move to the right towards  $B$ , then the ratio  $\frac{PA}{PB}$  continually increases until, when  $P$  approaches indefinitely near to  $B$  the ratio becomes infinite; and for intermediate positions it has passed continuously through every value between 1 and  $\infty$  (infinity).

When  $P$  is at the right of  $B$  the ratio

$$\frac{PA}{PB} = \frac{PB + AB}{PB} = 1 + \frac{AB}{PB},$$

and is therefore greater than 1.

When  $PB$  is very small  $\frac{AB}{PB}$  is very large, and as  $PB$  increases  $\frac{AB}{PB}$  diminishes until it becomes indefinitely small, and therefore  $\frac{PA}{PB}$  becomes as nearly equal to 1 as we please, and has passed continuously through every value between  $\infty$  and 1.

Hence for any assigned value of the ratio greater than 1 there are two positions for  $P$ , one between  $O$  and  $B$ , and one to the right of  $B$ .

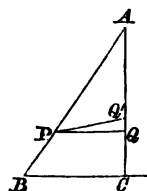
Similarly as  $P$  moves from  $O$  to  $A$ ,  $\frac{PA}{PB}$  passes through every value from 1 to 0, and as it moves to the left of  $A$  it

passes through every value from 0 to 1, and therefore for every value of the ratio less than 1 there are two positions for  $P$ , one between  $O$  and  $A$ , and one to the left of  $A$ .

### THEOREM 3.

*A straight line which divides the sides of a triangle proportionally is parallel to the base of the triangle.*

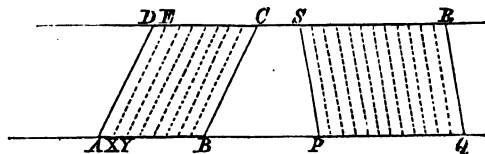
For let  $AP : PB :: AQ : QC$ , and suppose  $PQ$  not parallel to  $BC$ , but if possible let  $PQ'$  be parallel to  $BC$ ; then  $AP : PB :: AQ' : Q'C$ , and therefore  $AQ : QC :: AQ' : Q'C$ , which is impossible by Theorem 2.



### THEOREM 4.

*Parallelograms of the same altitude are to one another as their bases.*

Let  $ABCD$ ,  $PQRS$  be parallelograms of the same alti-



tude on the bases  $AB$ ,  $PQ$ .

Then shall  $DABC$  be to  $SPQR$  as  $AB$  to  $PQ$ .

Let  $AB, PQ$  be commensurable, and let them contain their common measure  $m$  and  $n$  times respectively. Through the points of division draw lines parallel to the sides of the parallelogram. Then the parallelograms will be divided into  $m$  and  $n$  equal parts respectively, (ii. 1. Cor. 2.)

and therefore  $DABC : SPQR :: m : n$   
 $:: AB : PQ.$

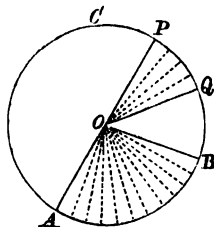
COR. 1. *Triangles of the same altitude are to one another as their bases.*

For a triangle is half the parallelogram on the same base and having the same altitude as the triangle.

#### THEOREM 5.

*In the same circle or in equal circles angles at the centre and sectors are to one another as the arcs on which they stand.*

Let  $ABC$  be a circle, of which  $O$  is the centre.



And let  $AOB, POQ$  be two angles at the centre.

Then  $\angle AOB : \angle POQ :: \text{arc } AB : \text{arc } PQ$   
 $:: \text{sector } AOB : \text{sector } POQ.$

Let the angles  $AOB$ ,  $POQ$  be commensurable, and let them contain their common measure  $m$  and  $n$  times respectively; and let the angles be divided into equal parts by radii.

Then (iii. 2) the areas and sectors are also divided into  $m$  and  $n$  equal parts respectively, and therefore

$$\begin{aligned}\text{arc } AB : \text{arc } PQ &:: m : n \\ &:: \angle AOB : \angle POQ \\ &:: \text{sector } AOB : \text{sector } POQ.\end{aligned}$$

## SECTION I.

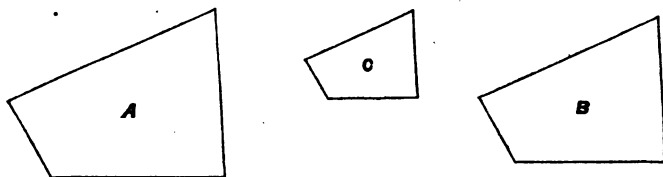
## SIMILAR FIGURES.

*Def. 1.* Similar rectilineal figures are those which have their angles equal, and the sides about the equal angles proportional.

*Def. 2.* Similar figures are said to be *similarly described upon given straight lines*, when those straight lines are homologous sides of the figures.

## THEOREM I.

*Rectilineal figures that are similar to the same rectilineal figure are similar to one another.*



*Proof.* Let  $A, B$  be each of them similar to  $C$ ;  
then will  $A$  be similar to  $B$ .

*Proof.* Since the angles of  $A$  and  $B$  are respectively equal to the angles of  $C$ ;



therefore also  $A$  and  $B$  are equiangular :

and since the sides about each angle of  $A$  are in the same ratio as the sides about the equal angle of  $C$ ; and the sides about each of  $B$  are also in the same ratio,

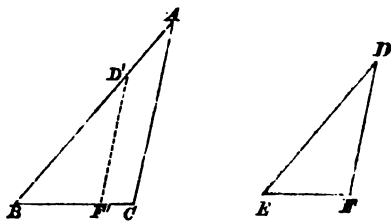
therefore the sides about the equal angles of  $A$  and  $B$  are proportionals;

therefore  $A$  is similar to  $B$ . (v. Def. 1.)

### THEOREM 2.

*If two triangles have their angles respectively equal, they are similar, and those sides which are opposite to the equal angles are homologous.*

Let  $ABC$ ,  $DEF$  be two triangles, which have the angles of the one equal to the angles of the other, viz.  $A$ ,  $B$ ,  $C$  respectively equal to  $D$ ,  $E$ ,  $F$  respectively;



Then shall they be similar, that is

$$AB : BC :: DE : EF,$$

and  $BC : CA :: EF : FD,$

and  $CA : AB :: FD : DE.$

Conceive the angle  $E$  placed on the angle  $B$ ; then  $F$  and  $D$  would fall as  $F'$  and  $D'$  on  $BC$  and  $BA$ , or on those lines produced: and because the  $\angle F =$  the  $\angle C$ , therefore  $F'D'$  is parallel to  $CA$ ;

and therefore  $BF' : BC :: BD' : BA$ , (IV. 1.)

and therefore  $BF' : BD' :: BC : BA$ ,

that is  $EF : ED :: BC : BA$ .

Similarly by placing  $F$  on  $C$ , and  $D$  on  $A$ , the other proportions are obtained; and therefore the triangles are similar.

This theorem is a generalization of Theorem 15 in Book 1. *If two angles and a side of one triangle are respectively equal to two angles and the corresponding side of another triangle, these triangles will be equal in all respects.*

### THEOREM 3.

*If two triangles have one angle of the one equal to one angle of the other and the sides about these angles proportional, they are similar, and those angles which are opposite to the homologous sides are equal.*

Let the triangles  $ABC$ ,  $DEF$  have the angles at  $B$  and  $E$  equal, and let  $BA : BC :: ED : EF$ , then will the triangles be similar.

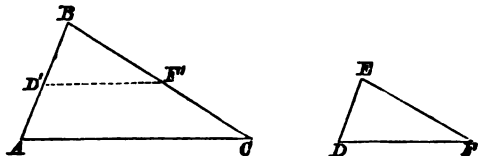
Conceive the angle  $E$  placed on the equal angle  $B$ , then  $D$  and  $F$  will fall as at  $D'$  and  $F'$  on the sides  $BA$ ,  $BC$ ,

and since  $BA : BC :: ED : EF$ ,

therefore  $BA : BD' :: BC : BF'$ ,

and therefore  $D'F'$  is parallel to  $AC$ , (IV. 3.)

and the angles  $BDF'$  and  $BFD'$ , that is,  $D$  and  $F'$  are



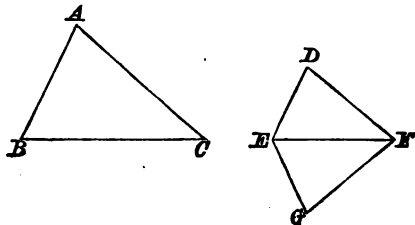
equal respectively to the angles  $A$  and  $C$ . Hence the triangles are equiangular and therefore similar.

This theorem is a generalization of Book I. Theorem 16. *If two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another, the triangles will be equal in all respects.*

#### THEOREM 4.

*If two triangles have the sides taken in order about each of their angles proportional, they are similar, and those angles which are opposite to the homologous sides are equal.*

Let  $ABC$ ,  $DEF$  be two triangles which have their sides



about each of their angles proportional,

that is,  $AB : BC :: DE : EF$ ,

and  $BC : CA :: EF : FD$ ,

and therefore also  $CA : AB :: FD : DE$ ;

then will the triangles  $ABC$ ,  $DEF$  be similar.

Conceive a triangle equiangular to  $ABC$  applied to  $EF$ , on the opposite side of the base  $EF$ , so that the angles  $FEG$ ,  $EFG$  are equal to  $B$  and  $C$  respectively.

Then since the triangle  $GEF$  is equiangular to  $ABC$ , it is therefore similar,

and therefore  $GE : EF :: AB : BC$ ;

but  $AB : BC :: DE : EF$ ;

and therefore  $GE : EF :: DE : EF$ ;

and therefore  $GE = ED$ .

Similarly  $GF = DF$ ,

and the triangle  $DEF$  is therefore equiangular to  $GEF$ , (I. 18) and therefore also to  $ABC$ .

Therefore the triangle  $DEF$  is similar to the triangle  $ABC$ .

This theorem is a generalization of Book I. Theorem 18. *If the three sides of one triangle are respectively equal to the three sides of another, these triangles will be equal in all respects.*

#### THEOREM 5.

*If two triangles have one angle of the one equal to one angle of the other, and the sides about one other angle in each proportional, so that the sides opposite the equal angles are homologous, the triangles have their third angles either equal or supplementary and in the former case the triangles are similar.*

Let  $ABC$ ,  $DEF$  be the two triangles having the angle  $B =$  the angle  $E$ ,

and  $BA : AC :: ED : DF$ ;

then will the angle  $C$  be equal or supplementary to the angle  $F$ .

*Proof.* The angle  $A$  is equal or unequal to the angle  $D$ .

If  $A = D$  (fig. 1), then, by Th. 3, the triangles are similar, and the angle  $C$  is equal to the angle  $F$ .

If  $A$  is not equal to  $D$ , as in fig. 2, at the point  $D$  make the angle  $EDG = BAC$ ;

Fig. (1.)



Fig. (2.)



then the triangle  $DEG$  is equiangular and similar to the triangle  $ABC$ ,

and therefore  $ED : DG = BA : AC$

But  $ED : DF = EA : AC$ ,

and therefore  $DG = DF$ ,

and therefore the angle  $DGF =$  the angle  $DFG$ ;

and because the angle  $EGD =$  the angle  $C$ ,

and  $EGD$  is supplementary to  $DGF$ ;

therefore  $C$  is supplementary to  $DFE$ .

Hence the angle  $C$  is either equal or supplementary to the angle  $F$ , and in the former case the triangles are similar.

This theorem is a generalization of Book 1. Theorem 26.

COR. 1. *If the two angles given equal are right angles or obtuse angles, the remaining angles must be both acute, and therefore cannot be supplementary, and are therefore equal, and the triangles are similar.*

COR. 2. *If the angles opposite to the other two homologous sides are both acute or both obtuse, or if one of them is a right angle, then these angles must be equal, and the triangles are similar.*

COR. 3. *If the side opposite the given angle in each triangle is not less than the other given side, then the given angles must be not less than the third angles: therefore the third angles must be both acute; and therefore cannot be supplementary. They are therefore equal and the triangles are similar.*

#### THEOREM 6.

*If two similar rectilineal figures are placed so as to have their corresponding sides parallel, all the straight lines joining the angular points of the one to the corresponding angular points of the other are parallel or meet in a point; and the distances from that point along any straight line to the points where it meets corresponding sides of the figures are in the ratio of the corresponding sides of the figures.*

Let  $ABCD, EFGH$  be the two rectilinear figures.

Let  $AB, BC$  be two consecutive sides of the rectilinear figure  $ABCD$ , and  $EF, FG$  the corresponding sides of the

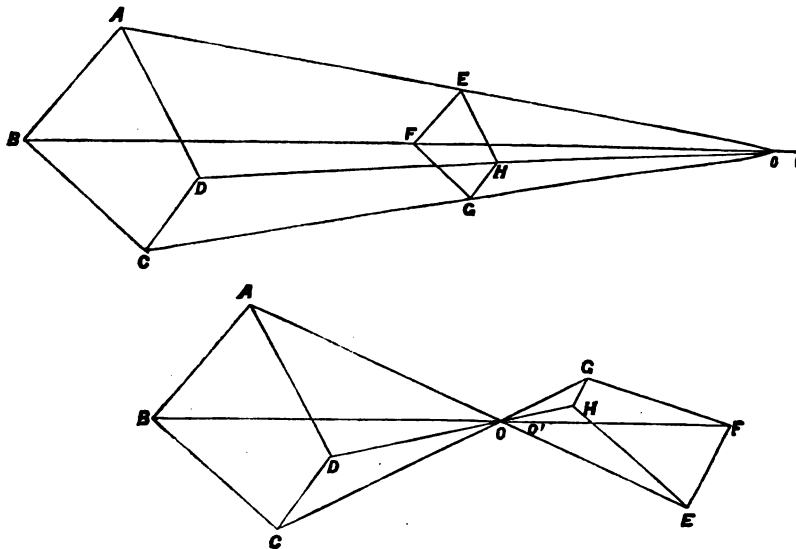


figure  $EFGH$ . Let  $AE, BF$  meet in  $O$ : it is required to prove that  $CG$  passes through  $O$ .

If not let it cut  $BF$  in some other point  $O'$ .

Then by the similar triangles  $ABO, EFO$ ,

$$AB : EF :: BO : FO ;$$

also by the similar triangles  $BCO', FGO'$ ,

$$BC : FG :: BO' : FO'.$$

But  $AB : EF :: BC : FG$

by hypothesis, since the rectilinear figures are similar ;

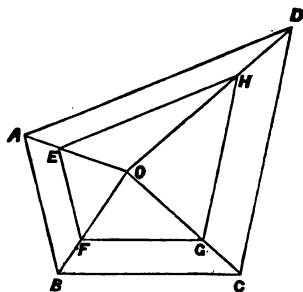
therefore  $BO : FO :: BO' : FO'$ ,

and therefore the points  $O, O'$  coincide.

Bk. iv. 2.

$\therefore CG$  does pass through  $O$ : and in the same manner  $DH$  passes through  $O$ .

**COR.** *Similar rectilineal figures may be divided into the same number of similar triangles.*



For if one rectilineal figure were placed within the other, and with their corresponding sides parallel, as in the figure, the lines  $AE, BF, CG, DH$  would all meet in one point  $O$ , and the triangles into which the polygons are respectively divided are similar.

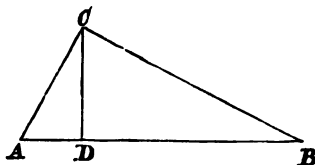
**Def. 3.** The point determined as in Theor. 6 is called a *centre of similarity* of the two rectilineal figures.

### THEOREM 7.

*In a right-angled triangle if a perpendicular is drawn from the right angle to the hypotenuse it divides the triangle into two other triangles which are similar to the whole and to one another.*



Let  $ACB$  be the triangle, right-angled at  $C$ ;  $CD$  the perpendicular.



Then the triangles  $CAD$ ,  $BAC$  have two angles  $CAD$  and  $CDA$  of the one equal respectively to  $BAC$ ,  $BCA$  of the other; therefore they are equiangular, and similar.

In the same manner  $DCB$  is equiangular and similar to either  $DAC$  or  $CAB$ .

Cor.  $AD : DC :: DC : DB$ ,

*or the perpendicular from the right angle of a right-angled triangle on the hypotenuse is a mean proportional between the segments of the base.*

Also  $BA : AC :: AC : AD$ ,

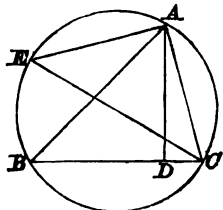
and  $AB : BC :: BC : BD$ ,

*or the side of a right-angled triangle is a mean proportional between the hypotenuse and the projection on it of that side.*

#### THEOREM 8.

*If from any angle of a triangle a straight line is drawn perpendicular to the base, the diameter of the circle circumscribing the triangle is a fourth proportional to the perpendicular and the sides of the triangle which contain that angle.*

Let  $ABC$  be a triangle, and let  $AD$  be drawn from the angle  $A$  perpendicular to the base  $BC$ ; and let  $CE$  be the diameter of the circle circumscribing the triangle  $ABC$ ;



then shall

$$AD : AB :: AC : CE.$$

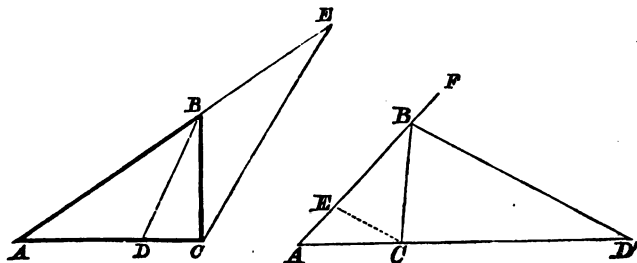
*Proof.* Join  $AE$ .

Then the triangles  $ADB$ ,  $CAE$  have the angles  $ABD = CEA$  in the same segment, and  $ADB = CAE$  being right angles; therefore they are similar, and therefore

$$AD : AB :: AC : CE.$$

#### THEOREM 9.

*If the interior or exterior vertical angle of a triangle is bisected by a straight line which also cuts the base, the base is divided internally or externally in the ratio of the sides of the triangle. And, conversely, if the base is divided internally or externally in the ratio of the sides of the triangle, the straight line drawn from the point of division to the vertex bisects the interior or exterior vertical angle.*



Let  $ABC$  be a triangle,  $BD$  the bisector of the angle  $ABC$ .

Then will  $AD : DC :: AB : BC$ .

Draw  $CE$  parallel to  $BD$  to meet  $AB$  produced.

Then by parallelism the angle  $BCE =$  the angle  $DBC$ , and the angle  $BEC =$  the angle  $ABD$  or  $FBD$ . But  $ABD$  or  $FBD = DBC$ , and therefore the angle  $BCE =$  the angle  $BEC$ ; and therefore  $BE = BC$ .

But because  $AE, AC$  are cut by the parallels  $DB, CE$ ; therefore  $AD : DC :: AB : BE$ , IV. 2.  
that is,  $AD : DC :: AB : BC$ .

COR. 1. *Conversely, if  $AD : DC :: AB : BC$ , then  $BD$  is the bisector of the angle  $ABC$ , or of  $CBF$ , according as  $AC$  is divided internally or externally in  $D$ .*

For there is only one internal bisector of the angle, and only one point  $D$  which divides the base internally or externally, so that

$$AD : DC :: AB : BC;$$

and therefore, since the bisector divides the base in this ratio, the line which divides the base in this ratio is the bisector.

(This may also be proved directly.)

COR. 2. *If  $AB = BC$ , then the ratio of  $AD' : D'C$  becomes  $= 1$ , which indicates that  $D'$  is at an infinite distance (by IV. 2). Hence the external bisector of the vertical angle of an isosceles triangle is parallel to the base.*

COR. 3. *If  $B$  moves so that the ratio  $AB : BC$  is constant, the bisectors of the interior and exterior angles will always pass through the fixed points  $D, D'$  which divide  $AC$  internally and externally in that ratio.*

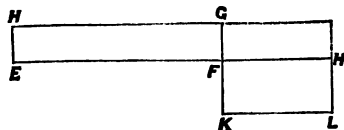
## SECTION II.

## AREAS.

## THEOREM 10.

*If four straight lines are proportional the rectangle contained by the extremes is equal to the rectangle contained by the means; and, conversely, if the rectangle contained by the extremes is equal to the rectangle contained by the means the four straight lines are proportional.*

$A$  —————  
 $B$  —————  
 $C$  —————  
 $D$  —————



Let  $A$  be to  $B$  as  $C$  to  $D$ ;  
 then will the rectangle contained by  $A$  and  $D$  be equal to  
 the rectangle contained by  $C$  and  $D$ .

*Proof.* Construct the rectangle  $EFGH$ , with the sides  $EF = A$ ,  $FG = D$ ; and also the rectangle  $HFKL$ , with the sides  $HF = B$ ,  $FK = C$ ; and place them so that  $EF$ ,  $FH$  are in one straight line, and therefore also  $GF$ ,  $FK$  in one straight line.

Complete the rectangle  $GH$ .

Then rectangle  $EG$  : rect.  $GH$  ::  $EF$  :  $FH$ ; (iv. 4)

::  $A$  :  $B$ ;

and  $\text{rect. } KH : \text{rect. } GH :: KF : GF;$   
 $:: C : D;$

but  $A : B :: C : D;$

therefore  $\text{rect. } FG : \text{rect. } GH :: \text{rect. } KH : \text{rect. } GH;$

therefore  $\text{rect. } EG = \text{rect. } GH;$

that is, the rectangle contained by  $A$  and  $D$  is equal to the rectangle contained by  $C$  and  $D$ .

Conversely, if the rectangle contained by  $A$  and  $D$  is equal to the rectangle contained by  $B$  and  $C$ ,

then  $A : B :: C : D.$

*Proof.* The same construction being made,  
 because  $\text{rect. } EG = \text{rect. } KH,$  (Hyp.)

therefore  $\text{rect. } EG : \text{rect. } GH :: \text{rect. } KH : \text{rect. } GH;$

but  $\text{rect. } EG : \text{rect. } GH :: EF : FH,$   
 $:: A : B,$

and  $\text{rect. } KH : \text{rect. } GH :: KF : FG,$   
 $:: C : D;$

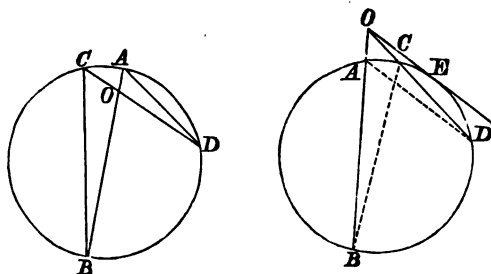
therefore  $A : B :: C : D.$

*COR.* If three straight lines are proportional the rectangle contained by the extremes is equal to the square on the mean; and, conversely, if the rectangle contained by the extremes of three straight lines is equal to the square on the mean the lines are proportional.

#### THEOREM II.

If two chords of a circle intersect either within or without a circle the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

Let  $AOB$ ,  $COD$  be the chords through  $O$ . Then is  $AO \times OB = CO \times OD$ .



For join  $CB$ ,  $AD$ . Then since the angle  $D$  = angle  $B$  in the same segment, and the angle at  $O$  common to the two triangles  $AOD$ ,  $BOC$ , the triangles are equiangular and similar,

$$\therefore AO : OD :: CO : OB,$$

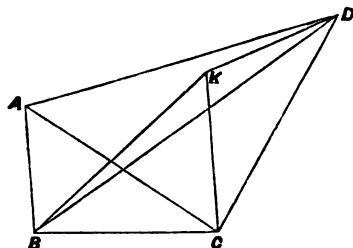
$$\therefore AO \times OB = CO \times OD.$$

**COR.** If one of the secants  $OCD$ , in figure 2, become a tangent, as  $OE$ , then  $OC$  and  $OD$  become equal to  $OE$ ; and therefore  $AO : OE :: OE : OB$ , and  $OE^2 = AO \times OB$ , or the square on the tangent to a circle from any point is equal to the rectangle contained by the intercepts on the secant drawn from that point.

#### THEOREM 12.

*The rectangle contained by the diagonals of a quadrilateral is less than the sum of the rectangles contained by opposite sides unless a circle can be circumscribed about a quadrilateral, in which case it is equal to that sum.*

Let  $ABCD$  be a quadrilateral figure: then will the rectangle contained by the diagonals  $AC$ ,  $BD$  be less than the sum of the rectangles contained by  $AB$ ,  $CD$  and by  $AD$ ,  $CB$  respectively, unless a circle can be described about  $ABCD$ .



*Proof.* At the point  $C$  in the straight line  $CD$  make the angle  $DCK$  equal to the angle  $ACB$ , and therefore also  $BCK$  equal to  $ACD$ ; and at the point  $D$  make the angle  $CDK$  equal to the angle  $BAC$ .

Join  $BK$ .

Then the triangle  $CDK$  is similar to the triangle  $CAB$  by construction;

and therefore  $AB : AC :: DK : DC$ ;

therefore the rectangle  $AB \times DC = \text{rect. } AC \times DK$ .

Again, because the triangles  $CDK$ ,  $CAB$  are similar,

$$BC : CA :: KC : CD,$$

and therefore  $BC : KC :: CA : CD$ ;

and the angle  $BCK$  is equal to the angle  $ACD$ , (Constr.)

therefore the triangles  $BCK$ ,  $ACD$  are similar, (Th. 3.)

and  $BC : BK :: AC : AD$ ;

therefore the rect.  $BC \times AD = \text{rect. } AC \times BK$ ;

but it was proved that the rect.  $AB \times DC = \text{rect. } CA \times DK$ ;  
therefore the sum of the rectangles  $BC \times AD + AB \times DC$   
= rectangle contained by  $AC$  and the sum of  $KB$  and  $KD$ .

But the sum of  $KB$  and  $KD$  is greater than  $BD$ ;  
therefore  $BC \times AD + AB \times DC$  is greater than  $CA \times BD$ .

But if the quadrilateral  $ABCD$  could have a circle described about it,

then the angle  $CDB$  would be equal to the angle  $CAB$  in the same segment ;

and therefore the point  $K$  would fall on  $BD$  ;

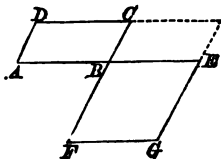
and therefore  $BK + KD = BD$ .

In this case therefore the rectangle contained by the diagonals is equal to the sum of the rectangles contained by the opposite sides of the quadrilateral.

### THEOREM 13.

*If two triangles or parallelograms have one angle of the one equal to one angle of the other, their areas have to one another the ratio compounded of the ratios of the including sides of the first to the including sides of the second.*

Let  $ABCD$ ,  $EBFG$  be the parallelograms, and let them be placed so as to have  $AB$ ,  $BE$  in one straight line, and therefore also, since the parallelograms are equiangular, so as to have  $CB$ ,  $BF$  in one straight line.





Complete the parallelogram  $CBE$ .

Then the ratio of  $DB : BG$  is compounded of the ratios of  $DB : CE$  and of  $CE$  to  $BG$ .

But  $DB : CE :: AB : BE$ ,

and  $CE : BG :: CB : BF$ ;

therefore, the ratio of  $DB : BG$  is compounded of the ratios  $AB : BE$  and  $CB : BF$ .

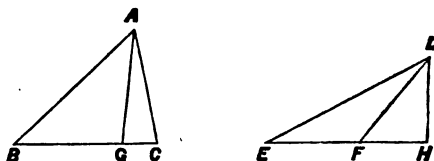
**COR. 1.** *If two triangles or parallelograms have one angle of the one supplementary to one angle of the other, their areas have to one another the ratio compounded of the ratios of the including sides of the first to the including sides of the second.*

**COR. 2.** *The ratio compounded of two ratios between straight lines is the same as the ratio of the rectangle contained by the antecedents to the rectangle contained by the consequents.*

#### THEOREM 14.

*Triangles and parallelograms have to one another the ratio compounded of the ratios of their bases and of their altitudes.*

Let  $ABC$ ,  $DEF$  be two triangles, having the altitudes  $AG$ ,  $DH$  respectively;



then shall the triangle  $ABC$  have to the triangle  $DEF$  the

ratio compounded of the ratios of the bases  $BC$  to  $EF$ , and of the altitudes  $AG$  to  $DH$ .

*Proof.* The triangle  $ABC$  is half the rectangle contained by  $AG$  and  $BC$ ; (11. Th. 2.)

and the triangle  $DEF$  is half the rectangle contained by  $DH$  and  $EF$ ;

therefore the triangle  $ABC$  : triangle  $DEF$   
 $\therefore$  rectangle  $AG, BC$  : rect.  $DH, EF$ ,

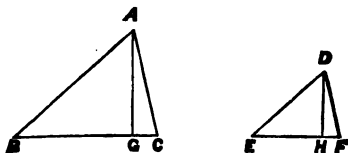
that is in the ratio compounded of  $AG : DH$  and of  $BC : EF$ . (Th. 13. Cor. 2.)

In the same manner it may be shewn that parallelograms are to one another in the ratio compounded of the ratios of their bases and of their altitudes.

#### THEOREM 15.

*Similar triangles are to one another in the duplicate ratio of their homologous sides.*

Let  $ABC, DEF$  be similar triangles having the angles at  $A, B, C$  respectively equal to the angles at  $D, E, F$ ;



then shall the triangles be to one another in the duplicate ratio of  $BC : EF$ .

*Proof.* Let fall the perpendiculars  $AG, DH$  to the sides  $BC, EF$ ,

then, by the last theorem,

w.

the triangle  $ABC$  is to the triangle  $DEF$  in the ratio compounded of the ratios of  $AG$  to  $DH$  and of  $BC$  to  $EF$ ;

but  $AG : DH :: AB : DE$ , by similar triangles,

and  $AB : DE :: BC : EF$ ; (Hyp.)

therefore  $AG : DH :: BC : EF$ ;

and therefore the ratio compounded of the ratios of  $AG : DH$  and of  $BC : EF$  is equal to the ratio compounded of the ratios of  $BC : EF$  and of  $BC : EF$ ;

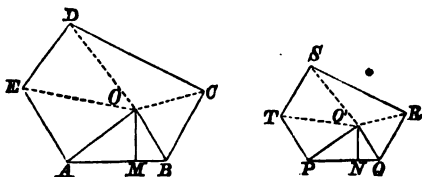
that is to the duplicate ratio of  $BC : EF$ ; (4 Def. 8.)

therefore the triangle  $ABC$  is to the triangle  $DEF$  in the duplicate ratio of  $BC$  to  $EF$ .

#### THEOREM 16.

*The areas of similar rectilineal figures are to one another in the duplicate ratio of their homologous sides.*

Let  $ABCDE$ ,  $PQRST$  be similar polygons.



Divide each of them into the same number of similar triangles by lines drawn from the points  $O$ ,  $O'$ . (v. 6. Cor.)

Let  $OAB$ ,  $O'PQ$  be two similar triangles.

Then the triangle  $OAB$  is to the triangle  $O'PQ$  in the duplicate ratio of  $AB : PQ$ ; (Th. 15.)

and the triangle  $AOE$  is to the triangle  $POR$  in the duplicate ratio of  $AE$  to  $PR$ :

but  $AB : PQ :: AE : PR$ ,

because the polygons are similar :

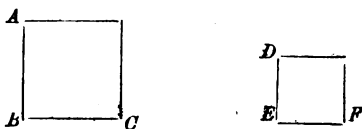
and therefore the triangle  $AOE$  is to the triangle  $O'PQ$  in the duplicate ratio of  $AB$  to  $PQ$ .

Similarly it may be proved that each of the triangles into which  $ABCDE$  is divided is to the corresponding triangle of those into which  $PQRST$  is divided in the duplicate ratio of  $AB : PQ$ .

Therefore the polygon  $ABCDE$  is to the polygon  $PQRST$  in the duplicate ratio of  $AB : PQ$ .

*COR. 1. Similar rectilineal figures are to one another as the squares described on their homologous sides.*

For if  $ABC$ ,  $DEF$  are squares, then by the theorem  $AC : DF$  is the duplicate ratio of  $BC : EF$ .



But any similar polygons similarly described on  $BC$  and  $EF$  are to one another in the duplicate ratio of  $BC$  to  $EF$ ; therefore they are in the ratio of the squares on  $BC$  and  $EF$ .

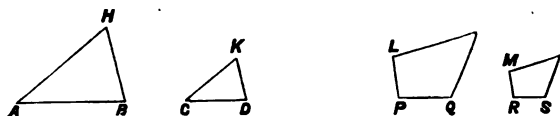
Hence, further, if three straight lines are in continued proportion, the first is to the third in the duplicate ratio of the first to the second;

but the squares on the first and second are in the duplicate ratio of the first to the second;

therefore the first is to the third, as the square described on the first is to the square described on the second:

and therefore, further, if three straight lines be in continued proportion, the 1st : 3rd as any polygon described on the 1st : the similar and similarly described polygon on the 2nd.

*COR. 2. If four straight lines are proportional and a pair of similar rectilineal figures are similarly described on the first and second, and also a pair on the third and fourth, these figures are proportional; and conversely, if a rectilineal figure on the first of four straight lines is to the similar and similarly described figure on the second as a rectilineal figure on the third is to the similar and similarly described figure on the fourth, the four straight lines are proportional.*



Let  $AB : CD :: PQ : RS$ ,  
and let the figures  $ABH$ ,  $CDK$ , and likewise  $LPQ$ ,  $MRS$ ,  
be respectively similar, and similarly situated on  $AB$ ,  $CD$ ,  
 $PQ$ ,  $RS$ ;

then  $ABH : CDK :: LPQ : MRS$ .

*Proof.* Since  $AB : CD :: PQ : RS$ ,  
therefore the duplicate ratio of  $AB : CD$  is equal to the  
duplicate ratio of  $PQ : RS$ .

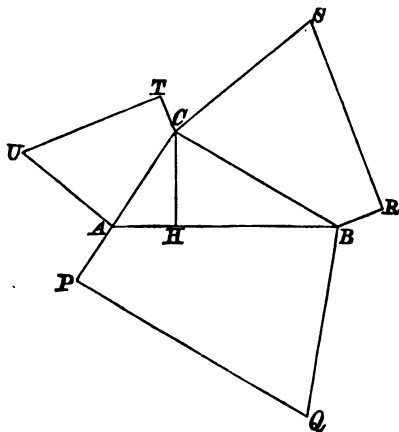
But  $HAB : KCD$  in the duplicate ratio of  $AB : CD$ ,  
and  $LPQ : MRS$  in the duplicate ratio of  $PQ : RS$ ;  
therefore  $HAB : KCD :: LPQ : MRS$ .

The converse follows by the rule of identity.

## THEOREM 17.

*In any right-angled triangle, any rectilineal figure described on the hypotenuse is equal to the sum of two similar and similarly described figures on the sides.*

Let  $ABC$  be a triangle right-angled at  $C$ ; and let  $APQB$ ,  $BRSC$ ,  $CTUA$  be similar figures similarly de-



scribed on the sides  $AB$ ,  $BC$ ,  $CA$ , that is, figures of which  $AB$ ,  $BC$ ,  $CA$  are homologous sides.

Then will  $APQB = BRSC + CTUA$ .

Draw  $CH$  perpendicular to  $AB$ .

Then  $AB : BC :: BC : BH$  by similar triangles,  
and therefore (v. 7. Cor.)

$APQB : BRSC :: AB : BH$  (v. 14. Cor. 3): in the same manner it may be shewn that

$$APQB : CTUA :: AB : AH,$$

and therefore

$$APQB : BRSC + CTUA :: AB : BH + AH;$$

but

$$AB = BH + AH;$$

and therefore  $APQB = QRSC + CTUA$ .

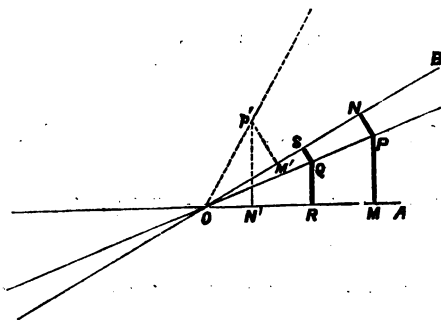
It is obvious that a special case of this theorem is the theorem proved before, that the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the sides.

## SECTION III.

## LOCI AND PROBLEMS.

## LOCI.

i. *The locus of a point whose distances from two fixed straight lines are in a constant ratio is a pair of straight lines, passing through the point of intersection of the given lines, if they intersect, and parallel to them, if the lines are parallel.*



First, let  $OA$ ,  $OB$  intersect in  $O$ , and let  $P$  be one of the points on the locus, so that  $PM : PN$  in the given ratio.

Join  $OP$ , and let  $Q$  be any point in  $OP$ .

Draw  $QR$ ,  $QS$  perpendicular to  $OA$ ,  $OB$ .

Then  $Q$  will be a point on the locus.

For by similar triangles  $OQR$ ,  $OPM$ ,

$$PM : QR :: OP : OQ,$$



and for the same reason

$$PN : QS :: OP : OQ;$$

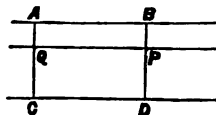
therefore  $PM : PN :: QR : QS,$

and therefore  $Q$  is a point on the locus; that is every point on a certain straight line through  $O$  satisfies the given condition.

Similarly there will be a line  $OP'$  dividing the angle at  $O$  supplementary to  $BOA$ .

Secondly, let the lines  $AB, CD$  be parallel.

Let  $P$  be a point on the locus, and let  $QP$  be parallel to  $AB$ , then  $QA : QC :: PB : PD$ , and therefore  $Q$  is a point on the locus;

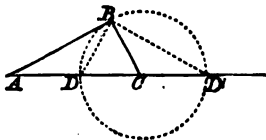


that is the locus consists of the line  $QP$  parallel to  $AB$  and  $CD$ .

Similarly there will be a line parallel to  $AB$  dividing  $BD$  externally in the same ratio.

ii. *The locus of a point whose distances from two fixed points are in a constant ratio (not one of equality) is a circle.*

Let  $A, C$  be the fixed points, and let  $B$  be one of the points on the locus, such that  $AB : BC$  in the given constant ratio.



Let  $D, D'$  divide  $AC$  internally and externally in the given ratio, so that

$$\begin{aligned} AD : DC &:: AD' : D'C \\ &:: AB : BC. \end{aligned}$$

Join  $DB, D'B$ .

Then, since  $AB : BC :: AD : DC$ ,

$DB$  bisects the angle  $ABC$ ; (Th. 9.)

and since  $AB : BC :: AD' : D'C$ ,

$D'B$  bisects the exterior angle at  $B$ ; (Th. 9.)

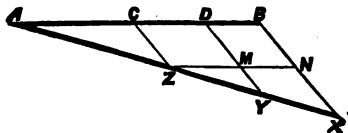
but the bisectors of adjacent supplementary angles are at right angles to one another;

therefore the angle  $DBD'$  is a right angle;

therefore the locus of  $B$  is the circle described on  $DD'$  as diameter.

#### PROBLEM I.

*To divide a straight line similarly to a given divided straight line.*



Let  $AB$  be the given divided line, and let it be required to divide  $AX$  similarly to  $AB$ .

*Construction.* Place  $AX$  so as to make an angle with  $AB$ .

Join  $BX$ ;

and through  $C, D$ , the points of division of  $AB$ ,

draw  $CZ, DY$  parallel to  $BX$ ;

then  $AX$  is divided similarly to  $AB$ .

*Proof.* Through  $Z$  draw  $ZMN$  parallel to  $AB$  to meet  $DY$ ,  $BX$  in  $M$ ,  $N$ .

Then because  $CZ$  is parallel to  $DY$ ,  
therefore  $AZ : ZY :: AC : CD$ ;

and because  $MY$  is parallel to  $NX$ ,  
therefore  $ZY : YX :: ZM : ZN$   
 $:: CD : DB$ ;

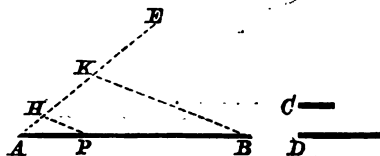
and therefore  $AX$  is divided similarly to  $AB$ .

### PROBLEM 2.

*To divide a straight line internally or externally in a given ratio\*.*

Let  $AB$  be the given line,  $C$  and  $D$  the lines which have the given ratio; then it is required to divide  $AB$  into two parts, which have to one another the ratio of  $C : D$ .

*Construction.* From  $A$  draw a line  $AE$  making any angle with  $AB$ , and cut off parts  $AH$ ,  $HK$  equal to  $C$  and



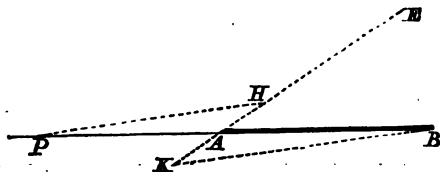
$D$  respectively. Join  $KB$ , and draw  $HP$  parallel to  $KB$ .  $P$  will be the point of division required.

\* By a *given ratio* is meant the ratio of two given lines, or of two given numbers: and since two lines can always be found which have the ratio of two given numbers, it follows that a given ratio can always be represented by the ratio of two given lines.

*Proof.* For since  $HP$  is parallel to  $KB$ ,  
 therefore  $AP : PB :: AH : HK$ ;  
 but  $AH = C$ ; and  $HK = D$ ;  
 therefore  $AP : PB :: C : D$ ,

that is,  $AB$  is divided into two parts which are to one another in the given ratio.

*Note.* This construction divides the line *internally* into parts which have the given ratio. If it is required to divide



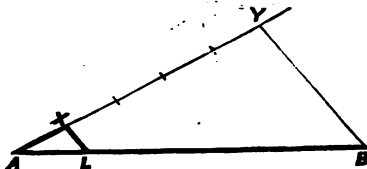
it *externally*,  $HK$  must be measured in the opposite direction along  $AE$ , as in the figure.

The proof will be the same as before.

### PROBLEM 3.

*From a given straight line to cut off any part required.*

Let  $AB$  be the given straight line : it is required to cut off from it any part required.



*Construction.* Draw any line  $AX$  making an angle with  $AB$ ;

produce  $AX$  indefinitely ; and cut off along it parts equal to

$AX$  until a length  $AY$  is obtained which is the same multiple of  $AX$  that  $AB$  is to be of the part required.

Join  $BY$ ;

and through  $X$  draw  $XL$  parallel to  $YB$ .

Then  $AL$  is the part required.

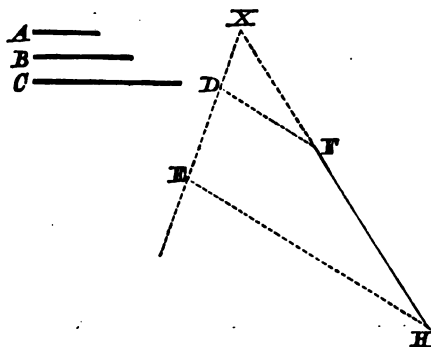
*Proof.* Because  $XL$  is parallel to  $YB$ ,  
therefore  $AL : AB :: AX : AY$ ;  
therefore  $AL$  is the same part of  $AB$  that  $AX$  is to  $AY$ ;  
that is,  $AL$  is the part required.

#### PROBLEM 4.

*To find a fourth proportional to three given straight lines.*

Let  $A, B, C$  be the given straight lines to which it is required to find a fourth proportional.

*Construction.* Take any angle  $X$ , and on one of its arms take  $XD, DE$  equal to  $A, B$  respectively: and on the other arm take  $XF$  equal to  $C$ . Join  $DF$ , and draw  $EH$  parallel to  $DF$ , to meet  $XF$  produced in  $H$ .



Then shall  $FH$  be the line required.

*Proof.* For since  $DF$  is parallel to  $EH$ ,

$$XD : DE :: XF : FH,$$

but  $XD$ ,  $DE$ , and  $XF$  are equal to  $A$ ,  $B$ ,  $C$  respectively;

therefore  $A : B :: C : FH$ ;

that is,  $FH$  is the fourth proportional required.

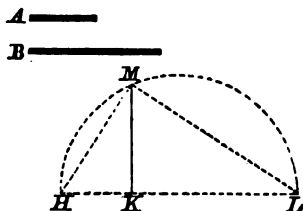
*COR.* Hence a third proportional to two given straight lines can be found, by taking  $C = B$ .

#### PROBLEM 5.

To find a mean proportional between two given straight lines.

Let  $A$ ,  $B$ , be the given straight lines: it is required to find a mean proportional between  $A$  and  $B$ .

*Construction.* Take  $HK$ ,  $KL$  in the same straight line, equal to  $A$  and  $B$  respectively. On  $HL$  describe a semi-



circle, and draw  $KM$  perpendicular to  $HL$  to meet the circumference in  $N$ .  $KM$  is the line required.

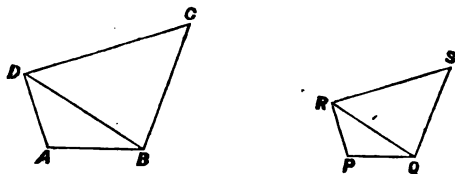
*Proof.* Join  $HM$ ,  $ML$ . Then since  $HML$  is a semi-circle,  $HML$  is a right angle; therefore  $MK$ , the perpen-

dicular from the right angle on the hypotenuse, is a mean proportional between the segments of the base ; (v. 7. Cor.) that is,  $MK$  is a mean proportional between  $HK$  and  $KL$ , or between  $A$  and  $B$ .

#### PROBLEM 6.

*On a straight line to describe a rectilinear figure similar to a given rectilinear figure.*

Let  $ABCD$  be the given rectilinear figure,  $PQ$  the given straight line: it is required to describe on  $PQ$  a rectilinear figure similar to  $ABCD$ .



*Construction.* Join  $DB$ : at  $P$ ,  $Q$  make angles  $QPR$ ,  $PQR$  equal respectively to  $BAD$ ,  $ABD$ : at  $R$ ,  $Q$ , in the straight line  $RQ$  make angles  $QRS$ ,  $RQS$  equal respectively to  $BDC$ ,  $DBC$ .

Then will the figure  $PQSR$  be similar to the figure  $ABCD$ .

*Proof.* By construction the angles of the figure  $PQSR$  are respectively equal to the angles of the figure  $ABCD$ : and by similar triangles  $ABD$ ,  $PQR$ ,

$$AB : BD :: PQ : QR;$$

again, by similar triangles  $DBC$ ,  $RQS$ ,

$$BD : BC :: QR : QS,$$

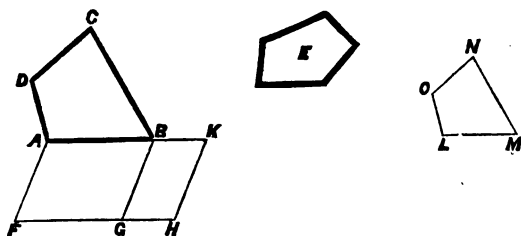
and therefore  $AB : BC :: PQ : QS$ .

In the same manner it may be shewn that the sides about each of the equal angles are proportionals :  
therefore the figure  $PQSR$  is similar to the figure  $ABCD$ .

## PROBLEM 7.

*To describe a rectilinear figure equal to one and similar to another given rectilinear figure.*

Let  $ABCD$  be the given rectilinear figure to which the required figure is to be similar ;  $E$  that to which it is to be equal.



*Construction.* On  $AB$  construct a parallelogram  $AFGB$  equal to  $ABCD$ ; (II. Prob. 2.)

On  $BG$  construct a parallelogram equal to  $E$ , and having an angle  $GBK =$  the angle  $FAB$ ; (II. Prob. 2.)  
between  $AB$  and  $BK$  find a mean proportional  $LM$ :  
(v. Prob. 5.)

On  $LM$  describe a rectilinear figure  $LMNO$  similar to  $ABCD$ , so that  $LM$  is homologous to  $AB$ . (v. Prob. 6.)

*Proof.* Since  $AB : LM :: LM : BK$ ,  
therefore  $AB : BK$  is the duplicate ratio of  $AB : LM$ .

And since  $AB : BK :: AG : BH$ ; (IV. 4.)



and  $ABCD : LMNO$  is the duplicate ratio of  $AB : LM$ ;

(v. 16. Cor. 1.)

therefore  $AG : BH :: ABCD : LMNO$ ;

but  $AG = ABCD$ , (Constr.)

therefore the figure  $LMNO$  = the parallelogram  $BH$ ;

but the parallelogram  $BH$  = the figure  $E$ , (Constr.)

therefore the figure  $LMNO$  is equal to the figure  $E$ ,

and it is also similar to the figure  $ABCD$ .

#### MISCELLANEOUS THEOREMS AND PROBLEMS.

1. The bisector of an angle of an equilateral triangle passes through one of the points of trisection of the perpendicular from either of the other angles on the opposite side.

2. The bisectors of the angles of a triangle intersect in one point.

3.  $ABC, PQR$  are two parallel lines such that

$$AB : BC :: PQ : QR:$$

prove that  $AP, BQ, CR$  are either parallel or meet in one point.

4. The external bisector of the vertical angle of an isosceles triangle is parallel to the base.

5. The line joining the middle points of the sides of a triangle is parallel to the base, and is equal to half the base.

6. The triangle formed by joining the middle points of the sides of a triangle is similar to the original triangle, and has one fourth of its area.

7. The lines that join the middle points of adjacent sides of a quadrilateral form a parallelogram. Under what circumstances will it be a rhombus, a square, or a rectangle?

8.  $ABC$  is a triangle, and in  $AC$  a point  $A'$  is taken, and  $BB'$  is cut off from  $CB$  produced, so that  $AA' = BB'$ . Prove that  $A'B$  is cut by  $AB$  into parts which have to one another the ratio  $CB : CA$ .

9. To inscribe a square in a triangle.

10. If two triangles are on equal bases between the same parallels any straight line parallel to their bases will cut off equivalent areas from the two triangles.

11. Make an equilateral triangle equivalent to a given square.

12. Find a point  $O$  within the triangle  $ABC$ , such that  $OAB$ ,  $OAC$ ,  $OBC$  shall be equivalent triangles.

13. The angle  $A$  of a triangle  $ABC$  is bisected by a line that meets the base in  $D$ :  $BC$  is bisected in  $O$ . Prove that  $OB : OD :: AB + AC : AB - AC$ .

14. Given the base, vertical angle, and ratio of the sides, construct the triangle.

15. Perpendiculars are drawn from any point within an equilateral triangle on the three sides; shew that their sum is invariable.

16. Deduce from Ptolemy's Theorem that if  $P$  is any point in the circumference of the circle circumscribing an equilateral triangle  $ABC$ , of the three lines  $PA$ ,  $PB$ ,  $PC$  one is equal to the sum of the other two.

17. From any point in the base of a triangle lines are drawn parallel to the two sides. Find the locus of the intersection of the diagonals of the parallelograms so formed.

18. In a quadrilateral figure which cannot be inscribed in a circle the rectangle contained by the diagonals is less than the sum of the rectangles contained by the opposite sides.

19.  $AB$  is a given line, and  $CD$  a given length on a line parallel to  $AB$ , and  $AC, BD$  intersect in  $O$ : prove that as  $CD$  varies in position, the locus of  $O$  is a line parallel to  $AB$ .

20.  $AB$  is a diameter of a circle of which  $AEF, BEG$  are chords.  $CED$  is drawn through  $E$  at right angles to  $AB$ : prove that  $CFDG$  is a quadrilateral such that the ratio of any pair of its adjacent sides is equal to the ratio of the other pair.

21. Divide a given arc of a circle into two parts which have their chords in a given ratio to one another.

22. If in two similar triangles lines are drawn from two of the equal angles to make equal angles with the homologous sides, these lines shall have to one another the same ratio as the sides of the triangle.

23. To make a rectilinear figure similar to a given rectilinear figure, and having a given ratio to it.

24. To find two straight lines which shall have the same ratio as two given rectangles.

25. To describe on a given straight line a rectangle equal to a given rectangle.

26. To make an isosceles triangle, with a given vertical angle, equal to a given triangle.

27. Let  $P, Q$  be points in  $AB$ , and  $AB$  produced, so that  $AP : PB :: AQ : QB$ , and let  $O$  be the middle point of  $PQ$ ; prove  $AO \times BO = OP^2$ .

28. In any triangle  $ABC$  the rectangle  $AB \times AC$  is equal to the rectangle contained by the diameter of the circle circumscribing the triangle, and the perpendicular from  $A$  on  $BC$ .

29. Hence shew that if  $A$  be the area of a triangle  $ABC$ ,  $D$  the diameter of the circumscribing circle,

$$A \times D = \frac{1}{2} AB \times BC \times CA.$$

30. Construct a rectangle equal to a given square, and having the sum of its adjacent sides equal to a given straight line.

31. Construct a rectangle equal to a given square, and having the difference of its adjacent sides equal to a given square.

32. Describe a rectangle equal to a given square, and having its sides in a given ratio.

33. If  $ABC$  is a triangle inscribed in a circle, and the tangent at  $A$  meets  $BC$  produced in  $D$ , prove that

$$CD : BD :: CA^2 : BA^2.$$

34.  $AB$  is a diameter of a circle, and at  $A$  and  $B$  tangents are drawn to the circle. If  $PCQ$  be a tangent at any point  $C$ , cutting the tangents at  $A, B$  in  $P, Q$ , prove that the radius of the circle is a mean proportional between the segments  $PC, QC$ .

35. With the same figure prove that if  $AQ$ ,  $BP$  intersect in  $R$ , then  $CR$  is parallel to  $AP$  or  $BQ$ .

36. If two triangles  $AEF$ ,  $ABC$  have a common angle  $A$ , prove that

$$\text{triangle } AEF : \text{triangle } ABC = AE \cdot AF : AB \cdot AC.$$

37. Given two points in a terminated straight line, find a point in the straight line such that its distances from the extremities of the line are to one another in the same ratio as its distances from the fixed points.

38. Divide a given straight line into two parts such that their squares may have a given ratio to one another.

39.  $AB$  is divided in  $C$ ; shew that the perpendiculars from  $A$ ,  $B$  on any straight line through  $C$  have to one another a constant ratio.

40. From the obtuse angle of a triangle to draw a line to the base which shall be a mean proportional between the segments of the base.

41. Divide a given triangle into two parts which shall have to one another a given ratio by a line parallel to one of the sides.

42. If from any point in the circumference of a circle perpendiculars be drawn to the sides, or sides produced of an inscribed triangle, prove that the feet of these perpendiculars lie in one straight line.

43. If a line be divided into any two parts to find the locus of the point in which these parts subtend equal angles.

44. If two circles touch each other externally, and also touch a straight line, prove that the part of the line between the points of contact is a mean proportional between the diameters of the circles.

45. Any regular polygon inscribed in a circle is a mean proportional between the inscribed and circumscribed regular polygons of half the number of sides.

46.  $ABC$  is a triangle, and  $O$  is the point of intersection of the perpendicular from  $A$  on the opposite side of the triangle: the circle which passes through the middle points of  $OA$ ,  $OB$ ,  $OC$ , will pass through the feet of the perpendiculars, and through the middle points of the sides of the triangle.

47. Describe a circle to touch a given straight line and a given circle, and to pass through a given point.

48.  $A$  and  $B$  are two points on the same side of a straight line which meet  $AB$  produced in  $C$ . Of all the points in this straight line find that at which  $AB$  subtends the greatest angle.

49. Inscribe a square in a given pentagon.

50.  $ABCD$  is a quadrilateral figure circumscribing a circle, and through the centre  $O$ , a line  $EOF$  equally inclined to  $AB$  and  $BC$  is drawn to meet them in  $E$  and  $F$ : prove that  $AE : EB :: CF : FD$ .



*By the same author.*

# SOLID GEOMETRY

AND

CONIC SECTIONS,

WITH APPENDICES ON TRANSVERSALS  
AND HARMONIC DIVISION.

FOR THE USE OF SCHOOLS.

*THIRD EDITION.*

*Extra fcap. 8vo., price 3s. 6d.*

MACMILLAN AND CO., LONDON.



NOTE BOOK ON PRACTICAL, SOLID, OR DESCRIPTIVE GEOMETRY. Containing Problems, with help for Solution. By J. H. EDGAR, M.A. and G. S. PRITCHARD. New Edition, revised and enlarged. Extra fcap. 8vo. 3s.

A GEOMETRICAL NOTE-BOOK, containing Easy Problems in Geometrical Drawing preparatory to the Study of Geometry. For the use of Schools. By F. E. KITCHENER, M.A., Mathematical Master at Rugby. New Edition. 4to. 2s.

SYLLABUS OF PLANE GEOMETRY. (Corresponding to Euclid, Books I.-VI.) Prepared by the Association for the Improvement of Geometrical Teaching. New Edition. Crown 8vo. 1s.

NATURAL GEOMETRY: an Introduction to the Logical Study of Mathematics. For Schools and Technical Classes. By A. MAULT. With Explanatory Models based upon the Tachymetrical works of Ed. LAGOUT. 18mo. 1s.  
Models to illustrate the above, in box, 12s. 6d.

ELEMENTS OF DESCRIPTIVE GEOMETRY. By J. B. MILLAR, C.E., Assistant Lecturer in Engineering in Owens College, Manchester. Crown 8vo. 6s.

MODERN METHODS IN ELEMENTARY GEOMETRY. By E. M. REYNOLDS, M.A., Mathematical Master in Clifton College. Crown 8vo. 3s. 6d.

MACMILLAN AND CO., LONDON.

July 1878.

A CATALOGUE  
OF  
EDUCATIONAL BOOKS,  
PUBLISHED BY  
MACMILLAN AND CO.,  
BEDFORD STREET, STRAND, LONDON.

CLASSICAL.

**ÆSCHYLUS**—*THE EUMENIDES*. The Greek Text, with Introduction, English Notes, and Verse Translation. By BERNARD DRAKE, M.A., late Fellow of King's College, Cambridge. 8vo. 3s. 6d.

**ARISTOTLE**—*AN INTRODUCTION TO ARISTOTLE'S RHETORIC*. With Analysis, Notes and Appendices. By E. M. COPE, Fellow and Tutor of Trinity College, Cambridge, 8vo. 14s.

*ARISTOTLE ON FALLACIES; OR, THE SOPHISTIC ELENCHI*. With Translation and Notes by E. POSTE, M.A. Fellow of Oriel College, Oxford. 8vo. 8s. 6d.

**ARISTOPHANES**—*THE BIRDS*. Translated into English Verse, with Introduction, Notes, and Appendices, by B. H. KENNEDY, D.D., Regius Professor of Greek in the University of Cambridge. Crown 8vo. 6s.

**BELCHER**—*SHORT EXERCISES IN LATIN PROSE COMPOSITION AND EXAMINATION PAPERS IN LATIN GRAMMAR*, to which is prefixed a Chapter on Analysis of Sentences. By the Rev. H. BELCHER, M.A., Assistant Master in King's College School, London. New Edition. 18mo. 1s. 6d. Key, 1s. 6d.

**BLACKIE**—*GREEK AND ENGLISH DIALOGUES FOR USE IN SCHOOLS AND COLLEGES*. By JOHN STUART BLACKIE, Professor of Greek in the University of Edinburgh. Second Edition. Fcap. 8vo. 2s. 6d.

10,000. 7. 78.

**CICERO**—*THE SECOND PHILIPPIC ORATION.* With Introduction and Notes. From the German of KARL HALM. Edited, with Corrections and Additions, by Professor JOHN E. B. MAYOR, M.A. Fellow and Classical Lecturer of St. John's College, Cambridge. Fourth Edition, revised. Fcap. 8vo. 5s.

*THE ORATIONS OF CICERO AGAINST CATILINA.* With Notes and an Introduction from the German of KARL HALM, with additions by Professor A. S. WILKINS, M.A., Owens College, Manchester. Fourth Edition. Fcap. 8vo. 3s. 6d.

*THE ACADEMICA OF CICERO.* The Text revised and explained by JAMES REID, M.A., Assistant Tutor and late Fellow of Christ's College, Cambridge. Fcap. 8vo. 4s. 6d.

*CICERO'S LETTERS.* Translated by G. E. JEANS, Assistant-Master at Haileybury. [Shortly.]

**DEMOSTHENES**—*ON THE CROWN*, to which is prefixed *ÆSCHINES AGAINST CTESIPHON.* The Greek Text with English Notes. By B. DRAKE, M.A., late Fellow of King's College, Cambridge. Fifth Edition. Fcap. 8vo. 5s.

**ELLIS**—*PRACTICAL HINTS ON THE QUANTITATIVE PRONUNCIATION OF LATIN*, for the use of Classical Teachers and Linguists. By A. J. ELLIS, B.A., F.R.S. Extra fcap. 8vo. 4s. 6d.

**GOODWIN**—*SYNTAX OF THE MOODS AND TENSES OF THE GREEK VERB.* By W. W. GOODWIN, Ph.D. New Edition, revised. Crown 8vo. 6s. 6d.

**GREENWOOD**—*THE ELEMENTS OF GREEK GRAMMAR*, including Accidence, Irregular Verbs, and Principles of Derivation and Composition; adapted to the System of Crude Forms. By J. G. GREENWOOD, Principal of Owens College, Manchester. Fifth Edition. Crown 8vo. 5s. 6d.

**HODGSON**—*MYTHOLOGY FOR LATIN VERSIFICATION.* A brief Sketch of the Fables of the Ancients, prepared to be rendered into Latin Verse for Schools. By F. HODGSON, B.D., late Provost of Eton. Fourth Edition, revised by F. C. HODGSON, M.A. 18mo. 3s.

**HOMERIC DICTIONARY.** For Use in Schools and Colleges. Translated from the German of Dr. G. Autenreith, with Additions and Corrections by R. P. KEEP, Ph.D. With numerous Illustrations. Crown 8vo. 6s.

**HOMER'S ODYSSEY—THE NARRATIVE OF ODYSSEUS.** With a Commentary by JOHN E. B. MAYOR, M.A., Kennedy Professor of Latin at Cambridge. Part I. Books IX.—XII. Fcap. 8vo. 3s.

**HORACE—THE WORKS OF HORACE,** rendered into English Prose, with Introductions, Running Analysis, and Notes, by J. LONSDALE, M.A., and S. LEE, M.A. Globe 8vo. 3s. 6d.

*THE ODES OF HORACE IN A METRICAL PARAPHRASE.* By R. M. HOVENDEN. Extra fcap. 8vo. 4s.

*HORACE'S LIFE AND CHARACTER.* An Epitome of his Satires and Epistles. By R. M. HOVENDEN. Extra fcap. 8vo. 4s. 6d.

*WORD FOR WORD FROM HORACE.* The Odes literally Versified. By W. T. THORNTON, C.B. Crown 8vo. 7s. 6d.

**JACKSON—FIRST STEPS TO GREEK PROSE COMPOSITION.** By BLOMFIELD JACKSON, M.A. Assistant-Master in King's College School, London. Fourth Edition, revised and enlarged. 18mo. 1s. 6d.

**JEBB—Works** by R. C. JEBB, M.A., Professor of Greek in the University of Glasgow.

*THE ATTIC ORATORS FROM ANTIPHON TO ISAEOS.* 2 vols. 8vo. 25s.

*THE CHARACTERS OF THEOPHRASTUS.* Translated from a Revised Text, with Introduction and Notes. Extra fcap. 8vo. 6s. 6d.

**JUVENAL—THIRTEEN SATIRES OF JUVENAL.** With a Commentary. By JOHN E. B. MAYOR, M.A., Kennedy Professor of Latin at Cambridge. Second Edition, enlarged. Vol. I. Crown 8vo. 7s. 6d. Or Parts I. and II. 3s. 6d. each.

**LIVY, Books XXI., XXII. HANNIBAL'S FIRST CAMPAIGN IN ITALY.** Edited, with Introduction and Notes, by the Rev. W. W. CAPES, Reader in Ancient History, Oxford. With Maps. Fcap. 8vo. [Shortly.]

**LYSIAS—SELECT ORATIONS.** Edited, with Notes, &c., by E. S. SHUCKBURGH. [In preparation.]

**MARSHALL — A TABLE OF IRREGULAR GREEK VERBS,** classified according to the arrangement of Curtius' Greek Grammar. By J. M. MARSHALL, M.A., one of the Masters in Clifton College. 8vo. cloth. Third Edition. 1s.

**MAYOR (JOHN E. B.)—FIRST GREEK READER.** Edited after KARL HALM, with Corrections and large Additions by Professor JOHN E. B. MAYOR, M.A., Fellow and Classical Lecturer of St. John's College, Cambridge. Third Edition, revised. Fcap. 8vo. 4s. 6d.

**BIBLIOGRAPHICAL CLUE TO LATIN LITERATURE.** Edited after HÜBNER, with large Additions by Professor JOHN E. B. MAYOR. Crown 8vo. 6s. 6d.

**MAYOR (JOSEPH B.)—GREEK FOR BEGINNERS.** By the Rev. J. B. MAYOR, M.A., Professor of Classical Literature in King's College, London. Part I., with Vocabulary, 1s. 6d. Parts II. and III., with Vocabulary and Index, 3s. 6d. complete in one Vol. New Edition. Fcap. 8vo. cloth. 4s. 6d.

**NIXON—PARALLEL EXTRACTS** arranged for translation into English and Latin, with Notes on Idioms. By J. E. NIXON, M.A., Classical Lecturer, King's College, London. Part I.—Historical and Epistolary. Second Edition, revised and enlarged. Crown 8vo. 3s. 6d.

**A FEW NOTES ON LATIN RHETORIC.** With Tables and Illustrations. By J. E. NIXON, M.A. Crown 8vo. 2s.

**PEILE (JOHN, M.A.)—AN INTRODUCTION TO GREEK AND LATIN ETYMOLOGY.** By JOHN PEILE, M.A., Fellow and Tutor of Christ's College, Cambridge, formerly Teacher of Sanskrit in the University of Cambridge. Third and Revised Edition. Crown 8vo. 10s. 6d.

**PLATO**—*THE REPUBLIC OF PLATO*. Translated into English, with an Analysis and Notes, by J. LL. DAVIES, M.A., and D. J. VAUGHAN, M.A. Third Edition, with Vignette Portraits of Plato and Socrates, engraved by JEENS from an Antique Gem. 18mo. 4s. 6d.

**PLAUTUS**—*THE MOSTELLARIA OF PLAUTUS*. With Notes, Prolegomena, and Excursus. By WILLIAM RAMSAY, M.A., formerly Professor of Humanity in the University of Glasgow. Edited by Professor GEORGE G. RAMSAY, M.A., of the University of Glasgow. 8vo. 14s.

**POTTS (A. W., M.A.)**—*HINTS TOWARDS LATIN PROSE COMPOSITION*. By ALEXANDER W. POTTS, M.A., LL.D., late Fellow of St. John's College, Cambridge; Assistant Master in Rugby School; and Head Master of the Fettes College, Edinburgh. New Edition. Extra fcap. 8vo. 3s.

**ROBY**—*A GRAMMAR OF THE LATIN LANGUAGE*, from Plautus to Suetonius. By H. J. ROBY, M.A., late Fellow of St. John's College, Cambridge. In Two Parts. Third Edition. Part I. containing :—Book I. Sounds. Book II. Inflexions. Book III. Word-formation. Appendices. Crown 8vo. 8s. 6d. Part II.—Syntax, Prepositions, &c. Crown 8vo. 10s. 6d.

"Marked by the clear and practised insight of a master in his art. A book that would do honour to any country."—*ATHENÆUM*.

**RUST**—*FIRST STEPS TO LATIN PROSE COMPOSITION*. By the Rev. G. RUST, M.A. of Pembroke College, Oxford, Master of the Lower School, King's College, London. Fifth Edition. 18mo. 1s. 6d.

**RUTHERFORD**—*A FIRST GREEK GRAMMAR*. By W. G. RUTHERFORD, M.A., Assistant Master of St. Paul's School, London. Extra fcap. 8vo. 1s.

**SALLUST**—*C. SALLUSTII CRISPI CATILINA ET JUGURTHA*. For use in Schools. With copious Notes. By C. MERIVALE, B.D. New Edition, carefully revised and enlarged. Fcap. 8vo. 4s. 6d. Or separately, 2s. 6d. each.

**TACITUS—THE HISTORY OF TACITUS TRANSLATED INTO ENGLISH.** By A. J. CHURCH, M.A., and W. J. BRODRIBB, M.A. With Notes and a Map. Third Edition. Crown 8vo. 6s.

"A scholarly and faithful translation."—SPECTATOR.

**THE AGRICOLA AND GERMANIA OF TACITUS.** A Revised Text, English Notes, and Maps. By A. J. CHURCH, M.A., and W. J. BRODRIBB, M.A. New Edition. Fcap. 8vo. 3s. 6d. Or separately, 2s. each.

"A model of careful editing, being at once compact, complete, and correct, as well as neatly printed and elegant in style."—ATHENÆUM.

**THE AGRICOLA AND GERMANY, WITH THE DIALOGUE ON ORATORY.** Translated into English by A. J. CHURCH, M.A., and W. J. BRODRIBB, M.A. With Maps and Notes. New and Revised Edition. Crown 8vo. 4s. 6d.

**THE ANNALS.** Translated, with Notes and Maps, by A. J. CHURCH and W. J. BRODRIBB. Second Edition. Crown 8vo. 7s. 6d.

**THE ANNALS.** Book VI. By the same Editors. With Notes. [Nearly ready.]

**TERENCE—HAUTON TIMORUMENOS.** Edited, with Introduction and Notes, by E. S. SHUCKBURGH, M.A. Fcap. 8vo. 3s. With Translation, 4s. 6d.

**THEOPHRASTUS—THE CHARACTERS OF THEOPHRASTUS.** An English Translation from a Revised Text. With Introduction and Notes. By R. C. JEBB, M.A., Professor of Greek in the University of Glasgow. Extra fcap. 8vo. 6s. 6d.

"A very handy and scholarly edition."—SATURDAY REVIEW.

**THRING—Works by the Rev. E. THRING, M.A., Head Master of Uppingham School.**

**A LATIN GRADUAL.** A First Latin Construing Book for Beginners. New Edition, enlarged, with Coloured Sentence Maps. Fcap. 8vo. 2s. 6d.

**A MANUAL OF MOOD CONSTRUCTIONS.** Fcap. 8vo. 1s. 6d.

**A CONSTRUING BOOK.** Fcap 8vo. 2s. 6d.

**THUCYDIDES**—*BOOKS VI. AND VII.*, with Notes. Fifth Edition, revised and enlarged, with Map. By the Rev. PERCIVAL FROST, M.A. Fcap. 8vo. 5s.

**VIRGIL**—*THE WORKS OF VIRGIL RENDERED INTO ENGLISH PROSE*, with Notes, Introductions, Running Analysis, and an Index, by JAMES LONSDALE, M.A., and SAMUEL LEE, M.A. Second Edition. Globe 8vo. 3s. 6d.; gilt edges, 4s. 6d.

"A more complete edition of Virgil in English it is scarcely possible to conceive than the scholarly work before us."—GLOBE.

**WRIGHT**—Works by J. WRIGHT, M.A., late Head Master of Sutton Coldfield School.

*HELLENICA; OR, A HISTORY OF GREECE IN GREEK*, as related by Diodorus and Thucydides; being a First Greek Reading Book, with explanatory Notes, Critical and Historical. Third Edition with a Vocabulary. Fcap. 8vo. 3s. 6d.

*A HELP TO LATIN GRAMMAR*; or, The Form and Use of Words in Latin, with Progressive Exercises. Crown 8vo. 4s. 6d.

*THE SEVEN KINGS OF ROME*. An Easy Narrative, abridged from the First Book of Livy by the omission of Difficult Passages; being a First Latin Reading Book, with Grammatical Notes. Fifth Edition. With Vocabulary, 3s. 6d.

*FIRST LATIN STEPS; OR, AN INTRODUCTION BY A SERIES OF EXAMPLES TO THE STUDY OF THE LATIN LANGUAGE*. Crown 8vo. 5s.

*ATTIC PRIMER*. Arranged for the Use of Beginners. Extra fcap. 8vo. 4s. 6d.

*A COMPLETE LATIN COURSE*, comprising Rules with Examples, Exercises, both Latin and English, on each Rule, and Vocabularies. Crown 8vo. 4s. 6d.

**XENOPHON**—The Hellenica of Xenophon. Books I. and II. Edited, with Introduction, Notes, and Map, by H. HAILSTONE, B.A. Fcap. 8vo. 4s. 6d.



# MATHEMATICS.

**AIRY**—Works by Sir G. B. AIRY, K.C.B., Astronomer Royal :—

*ELEMENTARY TREATISE ON PARTIAL DIFFERENTIAL EQUATIONS.* Designed for the Use of Students in the Universities. With Diagrams. Second Edition. Crown 8vo. 5s. 6d.

*ON THE ALGEBRAICAL AND NUMERICAL THEORY OF ERRORS OF OBSERVATIONS AND THE COMBINATION OF OBSERVATIONS.* Second Edition, revised. Crown 8vo. 6s. 6d.

*UNDULATORY THEORY OF OPTICS.* Designed for the Use of Students in the University. New Edition. Crown 8vo. 6s. 6d.

*ON SOUND AND ATMOSPHERIC VIBRATIONS.* With the Mathematical Elements of Music. Designed for the Use of Students in the University. Second Edition, Revised and Enlarged. Crown 8vo. 9s.

*A TREATISE OF MAGNETISM.* Designed for the Use of Students in the University. Crown 8vo. 9s. 6d.

**AIRY (OSMUND)**—*A TREATISE ON GEOMETRICAL OPTICS.* Adapted for the use of the Higher Classes in Schools. By OSMUND AIRY, B.A., one of the Mathematical Masters in Wellington College. Extra fcap. 8vo. 3s. 6d.

**BAYMA**—*THE ELEMENTS OF MOLECULAR MECHANICS.* By JOSEPH BAYMA, S.J., Professor of Philosophy, Stonyhurst College. Demy 8vo. 10s. 6d.

**BEASLEY**—*AN ELEMENTARY TREATISE ON PLANE TRIGONOMETRY*. With Examples. By R. D. BEASLEY, M.A., Head Master of Grantham Grammar School. Fifth Edition, revised and enlarged. Crown 8vo. 3s. 6d.

**BLACKBURN (HUGH)**—*ELEMENTS OF PLANE TRIGONOMETRY*, for the use of the Junior Class in Mathematics in the University of Glasgow. By HUGH BLACKBURN, M.A., Professor of Mathematics in the University of Glasgow. Globe 8vo. 1s. 6d.

**BOOLE**—Works by G. BOOLE, D.C.L., F.R.S., late Professor of Mathematics in the Queen's University, Ireland.

*A TREATISE ON DIFFERENTIAL EQUATIONS*. Third and Revised Edition. Edited by I. TODHUNTER. Crown 8vo. 14s.

*A TREATISE ON DIFFERENTIAL EQUATIONS*. Supplementary Volume. Edited by I. TODHUNTER. Crown 8vo. 8s. 6d.

*THE CALCULUS OF FINITE DIFFERENCES*. Crown 8vo. 10s. 6d. New Edition, revised by J. F. MOULTON.

**BROOK-SMITH (J.)**—*ARITHMETIC IN THEORY AND PRACTICE*. By J. BROOK-SMITH, M.A., LL.B., St. John's College, Cambridge; Barrister-at-Law; one of the Masters of Cheltenham College. New Edition, revised. Crown 8vo. 4s. 6d.

"A valuable Manual of Arithmetic of the Scientific kind. The best we have seen."—LITERARY CHURCHMAN.

**CAMBRIDGE SENATE-HOUSE PROBLEMS and RIDERS WITH SOLUTIONS** :—

1875—*PROBLEMS AND RIDERS*. By A. G. GREENHILL, M.A. Crown 8vo. 8s. 6d.

**CANDLER**—*HELP TO ARITHMETIC*. Designed for the use of Schools. By H. CANDLER, M.A., Mathematical Master of Uppingham School. Extra fcap. 8vo. 2s. 6d.

**CHEYNE**—*AN ELEMENTARY TREATISE ON THE PLANETARY THEORY.* By C. H. H. CHEYNE, M.A., F.R.A.S. With a Collection of Problems. Second Edition. Crown 8vo. 6s. 6d.

**CHRISTIE**—*A COLLECTION OF ELEMENTARY TEST-QUESTIONS IN PURE AND MIXED MATHEMATICS;* with Answers and Appendices on Synthetic Division, and on the Solution of Numerical Equations by Horner's Method. By JAMES R. CHRISTIE, F.R.S., Royal Military Academy, Woolwich. Crown 8vo. 8s. 6d.

**CLIFFORD**—*THE ELEMENTS OF DYNAMIC.* An Introduction to the Study of Motion and Rest in Solid and Fluid Bodies. By A. K. CLIFFORD, F.R.S., Professor of Applied Mathematics and Mechanics at University College, London. Part I. Crown 8vo. 7s. 6d.

**CUMMING**—*AN INTRODUCTION TO THE THEORY OF ELECTRICITY.* By LINNÆUS CUMMING, M.A., one of the Masters of Rugby School. With Illustrations. Crown 8vo. 8s. 6d.

**CUTHBERTSON**—*EUCLIDIAN GEOMETRY.* By FRANCIS CUTHBERTSON, M.A., LL.D., Head Mathematical Master of the City of London School. Extra fcap. 8vo. 4s. 6d.

**DALTON**—Works by the Rev. T. DALTON, M.A., Assistant Master of Eton College.

*RULES AND EXAMPLES IN ARITHMETIC.* New Edition. 18mo. 2s. 6d.

*Answers to the Examples are appended.*

*RULES AND EXAMPLES IN ALGEBRA.* Part I. Second Edition. 18mo. 2s. Part II. 18mo. 2s. 6d.

**DAY**—*PROPERTIES OF CONIC SECTIONS PROVED GEOMETRICALLY*. PART I., THE ELLIPSE, with Problems. By the Rev. H. G. DAY, M.A., Head Master of Sedburgh Grammar School. Crown 8vo. 3s. 6d.

**DREW**—*GEOMETRICAL TREATISE ON CONIC SECTIONS*. By W. H. DREW, M.A., St. John's College, Cambridge. Fifth Edition, enlarged. Crown 8vo. 5s.

*SOLUTIONS TO THE PROBLEMS IN DREW'S CONIC SECTIONS*. Crown 8vo. 4s. 6d.

**EDGAR (J. H.) and PRITCHARD (G. S.)**—*NOTE-BOOK ON PRACTICAL SOLID OR DESCRIPTIVE GEOMETRY*. Containing Problems with help for Solutions. By J. H. EDGAR, M.A., Lecturer on Mechanical Drawing at the Royal School of Mines, and G. S. PRITCHARD. Third Edition, revised and enlarged. Globe 8vo. 3s.

**FERRERS**—Works by the Rev. N. M. FERRERS, M.A., Fellow and Tutor of Gonville and Caius College, Cambridge.

*AN ELEMENTARY TREATISE ON TRILINEAR CO-ORDINATES*, the Method of Reciprocal Polars, and the Theory of Projectors. Third Edition, revised. Crown 8vo. 6s. 6d.

*AN ELEMENTARY TREATISE ON SPHERICAL HARMONICS, AND SUBJECTS CONNECTED WITH THEM*. Crown 8vo. 7s. 6d.

**FROST**—Works by PERCIVAL FROST, M.A., formerly Fellow of St. John's College, Cambridge; Mathematical Lecturer of King's College.

*AN ELEMENTARY TREATISE ON CURVE TRACING*. By PERCIVAL FROST, M.A. 8vo. 12s.

*SOLID GEOMETRY*. A New Edition, revised and enlarged of the Treatise by FROST and WOLSTENHOLME. In 2 Vols. Vol. I. 8vo. 16s.

**GODFRAY**—Works by HUGH GODFRAY, M.A., Mathematical Lecturer at Pembroke College, Cambridge.

*A TREATISE ON ASTRONOMY*, for the Use of Colleges and Schools. New Edition. 8vo. 12s. 6d.

*AN ELEMENTARY TREATISE ON THE LUNAR THEORY*, with a Brief Sketch of the Problem up to the time of Newton. Second Edition, revised. Crown 8vo. 5s. 6d.

**HEMMING**—*AN ELEMENTARY TREATISE ON THE DIFFERENTIAL AND INTEGRAL CALCULUS*, for the Use of Colleges and Schools. By G. W. HEMMING, M.A., Fellow of St. John's College, Cambridge. Second Edition, with Corrections and Additions. 8vo. 9s.

**JACKSON** — *GEOMETRICAL CONIC SECTIONS*. An Elementary Treatise in which the Conic Sections are defined as the Plane Sections of a Cone, and treated by the Method of Projection. By J. STUART JACKSON, M.A., late Fellow of Gonville and Caius College, Cambridge. Crown 8vo. 4s. 6d.

**JELLET (JOHN H.)**—*A TREATISE ON THE THEORY OF FRICTION*. By JOHN H. JELLET, B.D., Senior Fellow of Trinity College, Dublin; President of the Royal Irish Academy. 8vo. 8s. 6d.

**JONES and CHEYNE**—*ALGEBRAICAL EXERCISES*. Progressively Arranged. By the Rev. C. A. JONES, M.A., and C. H. CHEYNE, M.A., F.R.A.S., Mathematical Masters of Westminster School. New Edition. 18mo. 2s. 6d.

**KELLAND and TAIT**—*INTRODUCTION TO QUATERNIONS*, with numerous examples. By P. KELLAND, M.A., F.R.S.; and P. G. TAIT, M.A., Professors in the department of Mathematics in the University of Edinburgh. Crown 8vo. 7s. 6d.

**KITCHENER**—*A GEOMETRICAL NOTE-BOOK*, containing Easy Problems in Geometrical Drawing preparatory to the Study of Geometry. For the use of Schools. By F. E. KITCHENER, M.A., Mathematical Master at Rugby. Third Edition. 4to. 2s.

**MAULT**—*NATURAL GEOMETRY*: an Introduction to the Logical Study of Mathematics. For Schools and Technical Classes. With Explanatory Models, based upon the Tachymetrical Works of Ed. Lagout. By A. MAULT. 18mo. 1s. Models to Illustrate the above, in Box, 12s. 6d.

**MERRIMAN** — *ELEMENTS OF THE METHOD OF LEAST SQUARES*. By MANSFIELD MERRIMAN, Ph.D. Crown 8vo. 7s. 6d.

**MILLAR**—*ELEMENTS OF DESCRIPTIVE GEOMETRY*. By J. B. MILLAR, C.E., Assistant Lecturer in Engineering in Owens College, Manchester. Crown 8vo. 6s.

**MORGAN** — *A COLLECTION OF PROBLEMS AND EXAMPLES IN MATHEMATICS*. With Answers. By H. A. Morgan, M.A., Sadlerian and Mathematical Lecturer of Jesus College, Cambridge. Crown 8vo. 6s. 6d.

**NEWTON'S PRINCIPIA**. Edited by Prof. Sir W. THOMSON and Professor BLACKBURN. 4to. cloth. 31s. 6d.

"Undoubtedly the finest edition of the text of the 'Principia' which has hitherto appeared."—EDUCATIONAL TIMES.

*THE FIRST THREE SECTIONS OF NEWTON'S PRINCIPIA*, With Notes and Illustrations. Also a collection of Problems, principally intended as Examples of Newton's Methods. By PERCIVAL FROST, M.A. Third Edition. 8vo. 12s.

**PARKINSON**—Works by S. PARKINSON, D.D., F.R.S., Tutor and Prælector of St. John's College, Cambridge.

*AN ELEMENTARY TREATISE ON MECHANICS*. For the Use of the Junior Classes at the University and the Higher Classes in Schools. With a Collection of Examples. Fifth Edition, revised. Crown 8vo. cloth. 9s. 6d.

*A TREATISE ON OPTICS*. Third Edition, revised and enlarged. Crown 8vo. cloth. 10s. 6d.

**PHEAR**—*ELEMENTARY HYDROSTATICS*. With Numerous Examples. By J. B. PHEAR, M.A., Fellow and late Assistant Tutor of Clare College, Cambridge. Fourth Edition. Crown 8vo. cloth. 5s. 6d.

**PIRIE**—*LESSONS ON RIGID DYNAMICS*. By the Rev. G. PIRIE, M.A., Fellow and Tutor of Queen's College, Cambridge. Crown 8vo. 6s.

**PUCKLE**—*AN ELEMENTARY TREATISE ON CONIC SECTIONS AND ALGEBRAIC GEOMETRY*. With Numerous Examples and Hints for their Solution; especially designed for the Use of Beginners. By G. H. PUCKLE, M.A. Fourth Edition, revised and enlarged. Crown 8vo. 7s. 6d.

**RAWLINSON**—*ELEMENTARY STATICS*, by the Rev. GEORGE RAWLINSON, M.A. Edited by the Rev. EDWARD STURGES, M.A. Crown 8vo. 4s. 6d.

**RAYLEIGH**—*THE THEORY OF SOUND*. By LORD RAYLEIGH, M.A., F.R.S., formerly Fellow of Trinity College, Cambridge. In 2 Vols. 8vo. Vol. I. 12s. 6d. Vol. II. 12s. 6d.

**REYNOLDS**—*MODERN METHODS IN ELEMENTARY GEOMETRY*. By E. M. REYNOLDS, M.A., Mathematical Master in Clifton College. Crown 8vo. 3s. 6d.

**ROUTH**—Works by EDWARD JOHN ROUTH, M.A., F.R.S., late Fellow and Assistant Tutor of St. Peter's College, Cambridge; Examiner in the University of London.

*AN ELEMENTARY TREATISE ON THE DYNAMICS OF THE SYSTEM OF RIGID BODIES*. With numerous Examples. Third and enlarged Edition. 8vo. 21s.

*STABILITY OF A GIVEN STATE OF MOTION, PARTICULARLY STEADY MOTION*. Adams' Prize Essay for 1877. 8vo. 8s. 6d.

**SMITH**—Works by the Rev. BARNARD SMITH, M.A., Rector of Glaston, Rutland, late Fellow and Senior Bursar of St. Peter's College, Cambridge.

*ARITHMETIC AND ALGEBRA*, in their Principles and Application; with numerous systematically arranged Examples taken from the Cambridge Examination Papers, with especial

**SMITH** *Continued*—

reference to the Ordinary Examination for the B.A. Degree. Thirteenth Edition, carefully revised. Crown 8vo. 10s. 6d.

"To all those whose minds are sufficiently developed to comprehend the simplest mathematical reasoning, and who have not yet thoroughly mastered the principles of Arithmetic and Algebra, it is calculated to be of great advantage."—*ATHENÆUM*.

"Mr. Smith's work is a most useful publication. The rules are stated with great clearness. The examples are well selected, and worked out with just sufficient detail, without being encumbered by too minute explanations: and there prevails throughout it that just proportion of theory and practice which is the crowning excellence of an elementary work."—*DEAN PEACOCK*.

*ARITHMETIC FOR SCHOOLS.* New Edition. Crown 8vo. 4s. 6d.

"Admirably adapted for instruction, combining just sufficient theory with a large and well-selected collection of exercises for practice."—*JOURNAL OF EDUCATION*.

*A KEY TO THE ARITHMETIC FOR SCHOOLS.* New Edition. Crown 8vo. 8s. 6d.

*EXERCISES IN ARITHMETIC.* Crown 8vo. limp cloth. 2s. With Answers. 2s. 6d.

Or sold separately, Part I. 1s.; Part II. 1s.; Answers, 6d.

*SCHOOL CLASS-BOOK OF ARITHMETIC.* 18mo. cloth. 3s.

Or sold separately, Parts I. and II. 10d. each; Part III. 1s.

*KEYS TO SCHOOL CLASS-BOOK OF ARITHMETIC.* Parts I., II., and III., 2s. 6d. each.

*SHILLING BOOK OF ARITHMETIC FOR NATIONAL AND ELEMENTARY SCHOOLS.* 18mo. cloth. Or separately, Part I. 2d.; Part II. 3d.; Part III. 7d. Answers. 6d.

*THE SAME*, with Answers complete. 18mo, cloth. 1s. 6d.

*KEY TO SHILLING BOOK OF ARITHMETIC.* 18mo. 4s. 6d.

*EXAMINATION PAPERS IN ARITHMETIC.* 18mo. 1s. 6d. The same, with Answers, 18mo. 2s. Answers, 6d.



**SMITH** *Continued*—

*KEY TO EXAMINATION PAPERS IN ARITHMETIC.* 18mo. 4s. 6d.

*THE METRIC SYSTEM OF ARITHMETIC, ITS PRINCIPLES AND APPLICATIONS*, with numerous Examples, written expressly for Standard V. in National Schools. Third Edition. 18mo. cloth, sewed. 3d.

*A CHART OF THE METRIC SYSTEM*, on a Sheet, size 42 in. by 34 in. on Roller, mounted and varnished price 3s. 6d. Third Edition.

"We do not remember that ever we have seen teaching by a chart more happily carried out."—SCHOOL BOARD CHRONICLE.

Also a Small Chart on a Card, price 1d.

*EASY LESSONS IN ARITHMETIC*, combining Exercises in Reading, Writing, Spelling, and Dictation. Part I. for Standard I. in National Schools. Crown 8vo. 9d.

"We should strongly advise every one to study carefully Mr. Barnard Smith's Lessons in Arithmetic, Writing, and Spelling. A more excellent little work for a first introduction to knowledge cannot well be written. Mr. Smith's larger Text-books on Arithmetic and Algebra are already most favourably known, and he has proved now that the difficulty of writing a text-book which begins *ab ovo* is really surmountable; but we shall be much mistaken if this little book has not cost its author more thought and mental labour than any of his more elaborate text-books. The plan to combine arithmetical lessons with those in reading and spelling is perfectly novel, and it is worked out in accordance with the aims of our National Schools; and we are convinced that its general introduction in all elementary schools throughout the country will produce great educational advantages."—WESTMINSTER REVIEW.

*EXAMINATION CARDS IN ARITHMETIC.* (Dedicated to Lord Sandon.) With Answers and Hints.

Standards I. and II. in box, 1s. Standards III., IV. and V., in boxes, 1s. each. Standard VI. in Two Parts, in boxes, 1s. each.

A and B papers, of nearly the same difficulty, are given so as to prevent copying, and the Colours of the A and B papers differ in each Standard, and from those of every other Standard, so that a master or mistress can see at a glance whether the children have the proper papers.

**SNOWBALL**—*THE ELEMENTS OF PLANE AND SPHERICAL TRIGONOMETRY*; with the Construction and Use of Tables of Logarithms. By J. C. SNOWBALL, M.A. Eleventh Edition. Crown 8vo. 7s. 6d.

**SYLLABUS OF PLANE GEOMETRY** (corresponding to Euclid, Books I.—VI.). Prepared by the Association for the Improvement of Geometrical Teaching. Third Edition. Crown 8vo. 1s.

**TAIT and STEELE**—*A TREATISE ON DYNAMICS OF A PARTICLE*. With numerous Examples. By Professor TAIT and MR. STEELE. Fourth Edition, revised. Crown 8vo. 12s.

**TEBAY**—*ELEMENTARY MENSURATION FOR SCHOOLS*. With numerous Examples. By SEPTIMUS TEBAY, B.A., Head Master of Queen Elizabeth's Grammar School, Rivington. Extra fcap. 8vo. 3s. 6d.

**TODHUNTER**—Works by I. TODHUNTER, M.A., F.R.S., of St. John's College, Cambridge.

"Mr. Todhunter is chiefly known to students of Mathematics as the author of a series of admirable mathematical text-books, which possess the rare qualities of being clear in style and absolutely free from mistakes, typographical or other."—SATURDAY REVIEW.

*THE ELEMENTS OF EUCLID*. For the Use of Colleges and Schools. New Edition. 18mo. 3s. 6d.

*MENSURATION FOR BEGINNERS*. With numerous Examples. New Edition. 18mo. 2s. 6d.

*ALGEBRA FOR BEGINNERS*. With numerous Examples. New Edition. 18mo. 2s. 6d.

*KEY TO ALGEBRA FOR BEGINNERS*. Crown 8vo. 6s. 6d.

*TRIGONOMETRY FOR BEGINNERS*. With numerous Examples. New Edition. 18mo. 2s. 6d.

*KEY TO TRIGONOMETRY FOR BEGINNERS*. Crown 8vo. 8s. 6d.

**TODHUNTER** *Continued—*

*MECHANICS FOR BEGINNERS.* With numerous Examples. New Edition. 18mo. 4s. 6d.

*ALGEBRA.* For the Use of Colleges and Schools. Seventh Edition. Crown 8vo. 7s. 6d.

*KEY TO ALGEBRA FOR THE USE OF COLLEGES AND SCHOOLS.* Crown 8vo. 10s. 6d.

*AN ELEMENTARY TREATISE ON THE THEORY OF EQUATIONS.* Third Edition, revised. Crown 8vo. 7s. 6d.

*PLANE TRIGONOMETRY.* For Schools and Colleges. Sixth Edition. Crown 8vo. 5s.

*KEY TO PLANE TRIGONOMETRY.* Crown 8vo. 10s. 6d.

*A TREATISE ON SPHERICAL TRIGONOMETRY.* Third Edition, enlarged. Crown 8vo. 4s. 6d.

*PLANE CO-ORDINATE GEOMETRY,* as applied to the Straight Line and the Conic Sections. With numerous Examples. Fifth Edition, revised and enlarged. Crown 8vo. 7s. 6d.

*A TREATISE ON THE DIFFERENTIAL CALCULUS.* With numerous Examples. Seventh Edition. Crown 8vo. 10s. 6d.

*A TREATISE ON THE INTEGRAL CALCULUS AND ITS APPLICATIONS.* With numerous Examples. Fourth Edition, revised and enlarged. Crown 8vo. 10s. 6d.

*EXAMPLES OF ANALYTICAL GEOMETRY OF THREE DIMENSIONS.* Third Edition, revised. Crown 8vo. 4s.

*A TREATISE ON ANALYTICAL STATICS.* With numerous Examples. Fourth Edition, revised and enlarged. Crown 8vo. 10s. 6d.

**TODHUNTER** *Continued*—

*A HISTORY OF THE MATHEMATICAL THEORY OF PROBABILITY*, from the time of Pascal to that of Laplace. 8vo. 18s.

*RESEARCHES IN THE CALCULUS OF VARIATIONS*, principally on the Theory of Discontinuous Solutions: an Essay to which the Adams Prize was awarded in the University of Cambridge in 1871. 8vo. 6s.

*A HISTORY OF THE MATHEMATICAL THEORIES OF ATTRACTION, AND THE FIGURE OF THE EARTH*, from the time of Newton to that of Laplace. 2 vols. 8vo. 24s.

*AN ELEMENTARY TREATISE ON LAPLACE'S, LAME'S, AND BESSEL'S FUNCTIONS*. Crown 8vo. 10s. 6d.

**WILSON (J. M.)**—*ELEMENTARY GEOMETRY*. Books I. II. III. Containing the Subjects of Euclid's first Four Books. Following the Syllabus of the Geometrical Association. By J. M. WILSON, M.A., late Fellow of St. John's College Cambridge, and Mathematical Master of Rugby School. New Edition. Extra fcap. 8vo. 3s. 6d.

*SOLID GEOMETRY AND CONIC SECTIONS*. With Appendices on Transversals and Harmonic Division. For the Use of Schools. By J. M. WILSON, M.A. Third Edition. Extra fcap. 8vo. 3s. 6d.

**WILSON (W. P.)**—*A TREATISE ON DYNAMICS*. By W. P. WILSON, M.A., Fellow of St. John's College, Cambridge, and Professor of Mathematics in Queen's College, Belfast. 8vo. 9s. 6d.

**WOLSTENHOLME**—*A BOOK OF MATHEMATICAL PROBLEMS*, on Subjects included in the Cambridge Course. By JOSEPH WOLSTENHOLME, Fellow of Christ's College, sometime Fellow of St. John's College, and lately Lecturer in Mathematics at Christ's College. Crown 8vo. 8s. 6d.

"Judicious, symmetrical, and well arranged."—GUARDIAN.

## SCIENCE.

### ELEMENTARY CLASS-BOOKS.

#### **ASTRONOMY, by the Astronomer Royal.**

*POPULAR ASTRONOMY.* With Illustrations. By Sir G. B. AIRY, K.C.B., Astronomer Royal. New Edition. 18mo. 4s. 6d.

Six lectures, intended "to explain to intelligent persons the principles on which the instruments of an Observatory are constructed, and the principles on which the observations made with these instruments are treated for deduction of the distances and weights of the bodies of the Solar System."

#### **ASTRONOMY.**

*ELEMENTARY LESSONS IN ASTRONOMY.* With Coloured Diagram of the Spectra of the Sun, Stars, and Nebulæ, and numerous Illustrations. By J. NORMAN LOCKYER, F.R.S. New Edition. 18mo. 5s. 6d.

"Full, clear, sound, and worthy of attention, not only as a popular exposition, but as a scientific 'Index.'"—*ATHENÆUM*.

*QUESTIONS ON LOCKYER'S ELEMENTARY LESSONS IN ASTRONOMY.* For the Use of Schools. By JOHN FORBES-ROBERTSON. 18mo. cloth limp. 1s. 6d.

#### **PHYSIOLOGY.**

*LESSONS IN ELEMENTARY PHYSIOLOGY.* With numerous Illustrations. By T. H. HUXLEY, F.R.S., Professor of Natural History in the Royal School of Mines. New Edition. 18mo. 4s. 6d.

"Pure gold throughout."—*GUARDIAN*.

"Unquestionably the clearest and most complete elementary treatise on this subject that we possess in any language."—*WESTMINSTER REVIEW*.

**ELEMENTARY CLASS-BOOKS** *Continued—*

*QUESTIONS ON HUXLEY'S PHYSIOLOGY FOR SCHOOLS.* By T. ALCOCK, M.D. 18mo. 1s. 6d.

**BOTANY.**

*LESSONS IN ELEMENTARY BOTANY.* By D. OLIVER, F.R.S., F.L.S., Professor of Botany in University College, London. With nearly Two Hundred Illustrations. New Edition. 18mo. 4s. 6d.

**CHEMISTRY.**

*LESSONS IN ELEMENTARY CHEMISTRY, INORGANIC AND ORGANIC.* By HENRY E. ROSCOE, F.R.S., Professor of Chemistry in Owens College, Manchester. With numerous Illustrations and Chromo-Litho of the Solar Spectrum, and of the Alkalies and Alkaline Earths. New Edition. 18mo. 4s. 6d.

"As a standard general text-book it deserves to take a leading place."—SPECTATOR.

"We unhesitatingly pronounce it the best of all our elementary treatises on Chemistry."—MEDICAL TIMES.

*A SERIES OF CHEMICAL PROBLEMS*, prepared with Special Reference to the above, by T. E. Thorpe, Ph.D., Professor of Chemistry in the Yorkshire College of Science, Leeds. Adapted for the preparation of Students for the Government, Science, and Society of Arts Examinations. With a Preface by Professor ROSCOE. Fifth Edition, with Key, 18mo. 2s.

**POLITICAL ECONOMY.**

*POLITICAL ECONOMY FOR BEGINNERS.* By MILLICENT G. FAWCETT. New Edition. 18mo. 2s. 6d.

"Clear, compact, and comprehensive."—DAILY NEWS.

"The relations of capital and labour have never been more simply or more clearly expounded."—CONTEMPORARY REVIEW.

**LOGIC.**

*ELEMENTARY LESSONS IN LOGIC*; Deductive and Inductive, with copious Questions and Examples, and a Vocabulary of Logical Terms. By W. STANLEY JEVONS, M.A. Professor of Logic in University College, London. New Edition. 18mo. 3s. 6d.

"Nothing can be better for a school-book."—GUARDIAN.

"A manual alike simple, interesting, and scientific."—ATHENÆUM.

**ELEMENTARY CLASS-BOOKS** *Continued—*

**PHYSICS.**

*LESSONS IN ELEMENTARY PHYSICS.* By BALFOUR STEWART, F.R.S., Professor of Natural Philosophy in Owens College, Manchester. With numerous Illustrations and Chromolitho of the Spectra of the Sun, Stars, and Nebulae. New Edition. 18mo. 4s. 6d.

"The beau-ideal of a scientific text-book, clear, accurate, and thorough."  
EDUCATIONAL TIMES.

**PRACTICAL CHEMISTRY.**

*THE OWENS COLLEGE JUNIOR COURSE OF PRACTICAL CHEMISTRY.* By FRANCIS JONES, Chemical Master in the Grammar School, Manchester. With Preface by Professor ROSCOE, and Illustrations. New Edition. 18mo. 2s. 6d.

**ANATOMY.**

*LESSONS IN ELEMENTARY ANATOMY.* By ST. GEORGE MIVART, F.R.S., Lecturer in Comparative Anatomy at St. Mary's Hospital. With upwards of 400 Illustrations. 18mo. 6s. 6d.

"It may be questioned whether any other work on anatomy contains in like compass so proportionately great a mass of information."—LANCET.

"The work is excellent, and should be in the hands of every student of human anatomy."—MEDICAL TIMES.

**STEAM.**

*AN ELEMENTARY TREATISE.* By JOHN PERRY, Bachelor of Engineering, Whitworth Scholar, &c., late Lecturer in Physics at Clifton College. With numerous Woodcuts and Numerical Examples and Exercises. 18mo. 4s. 6d.

"The young engineer and those seeking for a comprehensive knowledge of the use, power, and economy of steam, could not have a more useful work, as it is very intelligible, well arranged, and practical throughout."—IRONMONGER.

**PHYSICAL GEOGRAPHY.**

*ELEMENTARY LESSONS IN PHYSICAL GEOGRAPHY.* By A. GEIKIE, F.R.S., Murchison Professor of Geology, &c., Edinburgh. With numerous Illustrations. 18mo. 4s. 6d.

*QUESTIONS ON THE SAME.* 1s. 6d.

**ELEMENTARY CLASS-BOOKS** *Continued*—**NATURAL PHILOSOPHY.**

**NATURAL PHILOSOPHY FOR BEGINNERS.** By I. TODHUNTER, M.A., F.R.S. Part I. The Properties of Solid and Fluid Bodies. 18mo. 3s. 6d.  
Part II. Sound, Light, and Heat. 18mo. 3s. 6d.

**MANUALS FOR STUDENTS.**

**FLOWER (W. H.)—AN INTRODUCTION TO THE OSTEOLOGY OF THE MAMMALIA.** Being the substance of the Course of Lectures delivered at the Royal College of Surgeons of England in 1870. By W. H. FLOWER, F.R.S., F.R.C.S., Hunterian Professor of Comparative Anatomy and Physiology. With numerous Illustrations. Second Edition enlarged. Crown 8vo. 10s. 6d.

**FOSTER and BALFOUR—THE ELEMENTS OF EMBRYOLOGY.** By MICHAEL FOSTER, M.D., F.R.S., and F. M. BALFOUR, M.A. Part I. crown 8vo. 7s. 6d.

**FOSTER and LANGLEY—A COURSE OF ELEMENTARY PRACTICAL PHYSIOLOGY.** By MICHAEL FOSTER, M.D., F.R.S., and J. N. LANGLEY, B.A. Third Edition. Crown 8vo. 6s.

**HOOKE (Dr.)—THE STUDENT'S FLORA OF THE BRITISH ISLANDS.** By Sir J. D. HOOKE, K.C.S.I., C.B., P.R.S., M.D., D.C.L. Second Edition, revised. Globe 8vo. 10s. 6d.

"Cannot fail to perfectly fulfil the purpose for which it is intended."—

LAND AND WATER.

"Certainly the fullest and most accurate manual of the kind that has yet appeared."—PALL MALL GAZETTE.

**HUXLEY and MARTIN—A COURSE OF PRACTICAL INSTRUCTION IN ELEMENTARY BIOLOGY.** By Professor HUXLEY, F.R.S., assisted by H. N. MARTIN, M.B., D.Sc. New Edition, revised. Crown 8vo. 6s.

"It is impossible for an intelligent youth, with this book in his hand, placing himself before any one of the organisms described, and carefully following the directions given, to fail to verify each point to which his attention is directed."—ATHENÆUM.



**HUXLEY—PHYSIOGRAPHY.** An Introduction to the Study of Nature. By Professor HUXLEY, F.R.S. With numerous Illustrations, and Coloured Plates. New Edition. Crown 8vo. 7s. 6d.

**OLIVER (Professor)—FIRST BOOK OF INDIAN BOTANY.** By DANIEL OLIVER, F.R.S., F.L.S., Keeper of the Herbarium and Library of the Royal Gardens, Kew, and Professor of Botany in University College, London. With numerous Illustrations. Extra fcap. 8vo. 6s. 6d.

"It contains a well-digested summary of all essential knowledge pertaining to Indian botany, wrought out in accordance with the best principles of scientific arrangement."—ALLEN'S INDIAN MAIL.

**PARKER and BETTANY—THE MORPHOLOGY OF THE SKULL.** By Professor PARKER and G. T. BETTANY. Illustrated. Crown 8vo. 10s. 6d.

Other volumes of these Manuals will follow.

#### NATURE · SERIES.

**THE SPECTROSCOPE AND ITS APPLICATIONS.** By J. NORMAN LOCKYER, F.R.S. With Coloured Plate and numerous Illustrations. Second Edition. Crown 8vo. 3s. 6d.

**THE ORIGIN AND METAMORPHOSES OF INSECTS.** By Sir JOHN LUBBOCK, M.P., F.R.S., D.C.L. With numerous Illustrations. Second Edition. Crown 8vo. 3s. 6d.

"We can most cordially recommend it to young naturalists."—*ATHENÆUM*.

**THE TRANSIT OF VENUS.** By G. FORBES, M.A., Professor of Natural Philosophy in the Andersonian University, Glasgow. Illustrated. Crown 8vo. 3s. 6d.

**THE COMMON FROG.** By ST. GEORGE MIVART, F.R.S., Lecturer in Comparative Anatomy at St. Mary's Hospital. With numerous Illustrations. Crown 8vo. 3s. 6d.

**POLARISATION OF LIGHT.** By W. SPOTTISWOODE, F.R.S. With many Illustrations. Second Edition. Crown 8vo. 3s. 6d.

**ON BRITISH WILD FLOWERS CONSIDERED IN RELATION TO INSECTS.** By Sir JOHN LUBBOCK, M.P., F.R.S. With numerous Illustrations. Second Edition. Crown 8vo. 4s. 6d.

**NATURE SERIES** *Continued*—**THE SCIENCE OF WEIGHING AND MEASURING, AND THE STANDARDS OF MEASURE AND WEIGHT.**

By H. W. CHISHOLM, Warden of the Standards. With numerous Illustrations. Crown 8vo. 4s. 6d.

**HOW TO DRAW A STRAIGHT LINE:** a Lecture on Linkages. By A. B. KEMPE. With Illustrations. Crown 8vo. 1s. 6d.

**LIGHT:** a Series of Simple, entertaining, and Inexpensive Experiments in the Phenomena of Light, for the Use of Students of every age. By A. M. MAYER and C. BARNARD. Crown 8vo, with numerous Illustrations. 2s. 6d.

*Other volumes to follow.*

**BALL (R. S., A.M.)—EXPERIMENTAL MECHANICS.** A Course of Lectures delivered at the Royal College of Science for Ireland. By R. S. BALL, A.M., Professor of Applied Mathematics and Mechanics in the Royal College of Science for Ireland. Royal 8vo. 16s.

**BLANFORD—THE RUDIMENTS OF PHYSICAL GEOGRAPHY FOR THE USE OF INDIAN SCHOOLS;** with a Glossary of Technical Terms employed. By H. F. BLANFORD, F.R.S. New Edition, with Illustrations. Globe 8vo. 2s. 6d.

**FLEISCHER—A SYSTEM OF VOLUMETRIC ANALYSIS.** Translated, with Notes and Additions, from the second German Edition, by M. M. PATTISON MUIR, F.R.S.E. With Illustrations. Crown 8vo. 7s. 6d.

**FOSTER—A TEXT BOOK OF PHYSIOLOGY.** By MICHAEL FOSTER, M.D., F.R.S. With Illustrations. New Edition, enlarged, with additional Illustrations and Plates. 8vo. 21s.

**GORDON—AN ELEMENTARY BOOK ON HEAT.** By J. E. H. GORDON, B.A., Gonville and Caius College, Cambridge. Crown 8vo. 2s.

**MIALl—STUDIES IN COMPARATIVE ANATOMY.** No. I.—The Skull of the Crocodile: a Manual for Students. By L. C. MIALl, Professor of Biology in the Yorkshire College and Curator of the Leeds Museum. 8vo. 2s. 6d.

**NEWCOMB—POPULAR ASTRONOMY.** By S. NEWCOMB, LL.D., Professor U.S. Naval Observatory. With 112 Illustrations and 5 Maps of the Stars. 8vo. 18s.

"It is unlike anything else of its kind, and will be of more use in circulating a knowledge of astronomy than nine-tenths of the books which have appeared on the subject of late years."—*Saturday Review*.

**REULEAUX — THE KINEMATICS OF MACHINERY.** Outlines of a Theory of Machines. By Professor F. REULEAUX. Translated and Edited by Professor A. B. KENNEDY, C.E. With 450 Illustrations. Medium 8vo. 21s.

**ROSCOE and SCHORLEMMER—CHEMISTRY,** A Complete Treatise on. By Professor H. E. ROSCOE, F.R.S., and Professor C. SCHORLEMMER, F.R.S. Vol. I.—The Non-Metallic Elements. With numerous Illustrations, and Portrait of Dalton. Medium 8vo. 21s. [Vol. II. in the press.]

**SHANN—AN ELEMENTARY TREATISE ON HEAT, IN RELATION TO STEAM AND THE STEAM-ENGINE.** By G. SHANN, M.A. With Illustrations. Crown 8vo. 4s. 6d.

**WRIGHT—METALS AND THEIR CHIEF INDUSTRIAL APPLICATIONS.** By C. ALDER WRIGHT, D.Sc., &c. Lecturer on Chemistry in St. Mary's Hospital Medical School. Extra fcap. 8vo. 3s. 6d.

## SCIENCE PRIMERS FOR ELEMENTARY SCHOOLS.

Under the joint Editorship of Professors HUXLEY, ROSCOE, and BALFOUR STEWART.

"These Primers are extremely simple and attractive, and thoroughly answer their purpose of just leading the young beginner up to the threshold of the long avenues in the Palace of Nature which these titles suggest."—*GUARDIAN*.

"They are wonderfully clear and lucid in their instruction, simple in style, and admirable in plan."—*EDUCATIONAL TIMES*.

**CHEMISTRY** — By H. E. ROSCOE, F.R.S., Professor of Chemistry in Owens College, Manchester. With numerous Illustrations. 18mo. 1s. New Edition. With Questions.

"A very model of perspicacity and accuracy."—*CHEMIST AND DRUGGIST*.

**SCIENCE PRIMERS** *Continued—*

**PHYSICS**—By BALFOUR STEWART, F.R.S., Professor of Natural Philosophy in Owens College, Manchester. With numerous Illustrations. 18mo. 1s. New Edition. With Questions.

**PHYSICAL GEOGRAPHY**—By ARCHIBALD GEIKIE, F.R.S., Murchison Professor of Geology and Mineralogy at Edinburgh. With numerous Illustrations. New Edition, with Questions. 18mo. 1s.

"Everyone of his lessons is marked by simplicity, clearness, and correctness."—*ATHENÆUM*.

**GEOLOGY**—By Professor GEIKIE, F.R.S. With numerous Illustrations. New Edition. 18mo. cloth. 1s.

"It is hardly possible for the dullest child to misunderstand the meaning of a classification of stones after Professor Geikie's explanation."—*SCHOOL BOARD CHRONICLE*.

**PHYSIOLOGY**—By MICHAEL FOSTER, M.D., F.R.S. With numerous Illustrations. New Edition. 18mo. 1s.

"The book seems to us to leave nothing to be desired as an elementary text-book."—*ACADEMY*.

**ASTRONOMY**—By J. NORMAN LOCKYER, F.R.S. With numerous Illustrations. New Edition. 18mo. 1s.

"This is altogether one of the most likely attempts we have ever seen to bring astronomy down to the capacity of the young child."—*SCHOOL BOARD CHRONICLE*.

**BOTANY**—By Sir J. D. HOOKER, K.C.S.I., C.B., President of the Royal Society. With numerous Illustrations. New Edition. 18mo. 1s.

"To teachers the Primer will be of inestimable value, and not only because of the simplicity of the language and the clearness with which the subject matter is treated, but also on account of its coming from the highest authority, and so furnishing positive information as to the most suitable methods of teaching the science of botany."—*NATURE*.

**LOGIC**—By Professor STANLEY JEVONS, F.R.S. New Edition. 18mo. 1s.

"It appears to us admirably adapted to serve both as an introduction to scientific reasoning, and as a guide to sound judgment and reasoning in the ordinary affairs of life."—*ACADEMY*.

**POLITICAL ECONOMY**—By Professor STANLEY JEVONS, F.R.S. 18mo. 1s.

"Unquestionably in every respect an admirable primer."—*SCHOOL BOARD CHRONICLE*.

*In preparation :—*

**INTRODUCTORY.** By Professor HUXLEY. &c. &c.

SCIENCE LECTURES AT SOUTH  
KENSINGTON.

*VOL. I.* Containing Lectures by Capt. ABNEY, Prof. STOKES, Prof. KENNEDY, F. G. BRAMWELL, Prof. G. FORBES, H. C. SORBY, J. T. BOTTOMLEY, S. H. VINES, and Prof. CAREY FOSTER. Crown 8vo. 6s.

*VOL. II.* In the Press.

Also, separately. 6d. each.

*SOUND AND MUSIC.* By Dr. W. H. STONE.

*PHOTOGRAPHY.* By Captain ABNEY, R.E.

*KINEMATIC MODELS.* By Professor KENNEDY, C.E.

*OUTLINES OF FIELD GEOLOGY.* By Professor GEIKIE, F.R.S.

*ABSORPTION OF LIGHT, AND FLUORESCENCE.*  
By Professor STOKES, F.R.S.

*TECHNICAL CHEMISTRY.* By Professor ROSCOE, F.R.S.

*THE STEAM ENGINE.* By F. J. BRAMWELL, C.E., F.R.S.

*ELECTROMETERS.* By J. BOTTOMLEY, F.R.S.E.

MANCHESTER SCIENCE LECTURES  
FOR THE PEOPLE.

Eighth Series, 1876-7. Crown 8vo. Illustrated. 6d. each.

*WHAT THE EARTH IS COMPOSED OF.* By Professor ROSCOE, F.R.S.

*THE SUCCESSION OF LIFE ON THE EARTH.* By Professor WILLIAMSON, F.R.S.

*WHY THE EARTH'S CHEMISTRY IS AS IT IS.* By J. N. LOCKYER, F.R.S.

Also complete in One Volume. Crown 8vo. cloth. 2s.

## MISCELLANEOUS.

**ABBOTT**—*A SHAKESPEARIAN GRAMMAR*. An Attempt to illustrate some of the Differences between Elizabethan and Modern English. By the Rev. E. A. ABBOTT, D.D., Head Master of the City of London School. New Edition. Extra fcap. 8vo. 6s.

"Valuable not only as an aid to the critical study of Shakespeare, but as tending to familiarise the reader with Elizabethan English in general."  
—*ATHENÆUM*.

**ANDERSON**—*LINEAR PERSPECTIVE, AND MODEL DRAWING*. A School and Art Class Manual, with Questions and Exercises for Examination, and Examples of Examination Papers. By LAURENCE ANDERSON. With Illustrations. Royal 8vo. 2s.

**BARKER**—*FIRST LESSONS IN THE PRINCIPLES OF COOKING*. By LADY BARKER. New Edition. 18mo. 1s.

"An unpretending but invaluable little work . . . The plan is admirable in its completeness and simplicity; it is hardly possible that anyone who can read at all can fail to understand the practical lessons on bread and beef, fish and vegetables."—*SPECTATOR*.

**BERNERS**—*FIRST LESSONS ON HEALTH*. By J. BERNERS. Seventh Edition. 18mo. 1s.

**BREYMANN**—Works by HERMANN BREYMANN, Ph.D., Professor of Philology in the University of Munich.

*A FRENCH GRAMMAR BASED ON PHILOLOGICAL PRINCIPLES*. Second Edition. Extra fcap. 8vo. 4s. 6d.

"A good, sound, valuable philological grammar."—*SCHOOL BOARD CHRONICLE*.

*FIRST FRENCH EXERCISE BOOK*. Extra fcap. 8vo. 4s. 6d.

*SECOND FRENCH EXERCISE BOOK*. Extra fcap. 8vo. 2s. 6d.

**CALDERWOOD**—*HANDBOOK OF MORAL PHILOSOPHY*.

By the Rev. HENRY CALDERWOOD, LL.D., Professor of Moral Philosophy, University of Edinburgh. Fourth Edition. Crown 8vo. 6s.

"A compact and useful work . . . will be an assistance to many students outside the author's own University."—*GUARDIAN*.

**DELAMOTTE—A BEGINNER'S DRAWING BOOK.** By P. H. DELAMOTTE, F.S.A. Progressively arranged. New Edition improved. Crown 8vo. 3s. 6d.

"A concise, simple, and thoroughly practical work."—GUARDIAN.

Now publishing, crown 8vo., 2s. 6d. each.

**ENGLISH MEN OF LETTERS.** Edited by JOHN MORLEY.

These short books are addressed to the general public, with a view both to stirring and satisfying an interest in literature and its great topics in the minds of those who have to run as they read. An immense class is growing up, and must every year increase, whose education will have made them alive to the importance of the masters of our literature, and capable of intelligent curiosity as to their performances. The series is intended to give the means of nourishing this curiosity to an extent that shall be copious enough to be profitable for knowledge and life, and yet be brief enough to serve those whose leisure is scanty.

The following are arranged for:—

<i>SPENSER</i>	.	.	.	THE DEAN OF ST. PAUL'S.	
<i>HUME</i>	.	.	.	PROFESSOR HUXLEY.	
<i>BUNYAN</i>	.	.	.	J. A. FROUDE	
<i>JOHNSON</i>	.	.	.	LESLIE STEPHEN.	[Ready.
<i>GOLDSMITH</i>	.	.	.	WILLIAM BLACK.	[In the Press.
<i>DICKENS</i>	.	.	.	T. HUGHES, Q.C.	
<i>MILTON</i>	.	.	.	MARK PATTISON.	
<i>WORDSWORTH</i>	.	.	.	GOLDWIN SMITH.	
<i>SWIFT</i>	.	.	.	JOHN MORLEY.	
<i>BURNS</i>	.	.	.	PRINCIPAL SHAIRP.	
<i>SCOTT</i>	.	.	.	R. H. HUTTON.	[Ready.
<i>SHELLEY</i>	.	.	.	J. A. SYMONDS.	
<i>GIBBON</i>	.	.	.	J. C. MORISON.	[Ready.
<i>BYRON</i>	.	.	.	PROFESSOR NICHOL.	
<i>DEFOE</i>	.	.	.	W. MINTO.	

*Others will follow.*

**FAWCETT**—*TALES IN POLITICAL ECONOMY.* By MILLICENT GARRETT FAWCETT. Globe 8vo. 3s.

"The idea is a good one, and it is quite wonderful what a mass of economic teaching the author manages to compress into a small space."—*ATHENÆUM*.

**FEARON**—*SCHOOL INSPECTION.* By D. R. FEARON, M.A., Assistant Commissioner of Endowed Schools. Third Edition. Crown 8vo. 2s. 6d.

"The work is admirably adapted to serve the purpose for which it has been written. It is calculated to be eminently useful, and to have a powerful influence for good on our elementary education."—*ATHENÆUM*.

**GLADSTONE**—*SPELLING REFORM FROM AN EDUCATIONAL POINT OF VIEW.* By J. H. GLADSTONE, E.R.S., Member for the School Board for London. New Edition. Crown 8vo. 1s. 6d.

**GOLDSMITH**—*THE TRAVELLER*, or a Prospect of Society; and *THE DESERTED VILLAGE.* By OLIVER GOLDSMITH. With Notes Philological and Explanatory, by J. W. HALES, M.A. Crown 8vo. 6d.

**HALES**—*LONGER ENGLISH POEMS*, with Notes, Philological and Explanatory, and an Introduction on the Teaching of English. Chiefly for Use in Schools. Edited by J. W. HALES, M.A., Professor of English Literature at King's College, London, &c. &c. Fifth Edition. Extra fcap. 8vo. 4s. 6d.

"The notes are very full and good, and the book, edited by one of our most cultivated English scholars, is probably the best volume of selections ever made for the use of English schools."—*PROFESSOR MORLEY'S First Sketch of English Literature.*

**HOLE**—*A GENEALOGICAL STEMMA OF THE KINGS OF ENGLAND AND FRANCE.* By the Rev. C. HOLE. On Sheet. 1s.

**JEPHSON**—*SHAKESPEARE'S "TEMPEST."* With Glossarial and Explanatory Notes. By the Rev. J. M. JEPHSON. Second Edition. 18mo. 1s.



**LITERATURE PRIMERS**—Edited by JOHN RICHARD GREEN,  
Author of "A Short History of the English People."

*ENGLISH GRAMMAR.* By the Rev. R. MORRIS, LL.D.,  
President of the Philological Society. New Edition. 18mo.  
cloth. 1s.

"A work quite precious in its way . . . . An excellent English Grammar for the lowest form."—EDUCATIONAL TIMES.

*THE CHILDREN'S TREASURY OF LYRICAL POETRY.* Selected and arranged with Notes by FRANCIS TURNER PALGRAVE. In Two Parts. 18mo. 1s. each.

*ENGLISH LITERATURE.* By the Rev. STOPFORD BROOKE, M.A. New Edition. 18mo. 1s.

"Unquestionably the best short sketch of English literature that has appeared."—ATHENÆUM.

*PHILOLOGY.* By J. PEILE, M.A. 18mo. 1s.

"Surely so much matter thoroughly good and clear was never before brought close together in the same compass."—SATURDAY REVIEW.

*GREEK LITERATURE.* By Professor JEBB, M.A. 18mo. 1s.

*SHAKSPERE.* By Professor DOWDEN. 18mo. 1s.

*ENGLISH GRAMMAR EXERCISES.* By R. MORRIS, LL.D., and H. C. BOWEN, M.A. 18mo. 1s.

*HOMER.* By the Right Hon. W. E. GLADSTONE, M.P. 18mo. 1s.

In preparation :—

*LATIN LITERATURE.*

*BIBLE PRIMER.* By the Rev. STOPFORD BROOKE.

*CHAUCER.* By F. J. FURNIVALL, M.A.

**MACMILLAN'S PROGRESSIVE FRENCH COURSE**—By  
G. EUGENE-FASNACHT, Senior Master of Modern Languages  
Harpur Foundation Modern School, Bedford.

I.—First year, containing Easy Lessons on the Regular 'Accidence. Extra fcap. 8vo. 1s.

II.—Second Year, containing Conversational Lessons on Systematic Accidence and Elementary Syntax. With Philological illustrations and Etymological Vocabulary. 1s. 6d.

**MACMILLAN'S PROGRESSIVE GERMAN COURSE**—By  
G. EUGENE FASNACHT.

Part I.—First Year. Easy Lessons and Rules on the Regular  
Accidence. Extra fcap. 8vo. 1s. 6d.

Part II.—Second Year. Conversational Lessons in Systematic  
Accidence and Elementary Syntax. With Philological Illustra-  
tions and Etymological Vocabulary. Extra fcap. 8vo. 2s.

**MARTIN** — *THE POET'S HOUR*: Poetry selected and  
arranged for Children. By FRANCES MARTIN. Third  
Edition. 18mo. 2s. 6d.

*SPRING-TIME WITH THE POETS*: Poetry selected by  
FRANCES MARTIN. Second Edition. 18mo. 3s. 6d.

**MASSON (GUSTAVE)**—*A COMPENDIOUS DICTIONARY  
OF THE FRENCH LANGUAGE* (French-English and  
English-French). Followed by a List of the Principal Di-  
verging Derivations, and preceded by Chronological and  
Historical Tables. By GUSTAVE MASSON, Assistant-Master  
and Librarian, Harrow School. Fourth Edition. Crown 8vo.  
half-bound. 6s.

"A book which any student, whatever may be the degree of his ad-  
vancement in the language, would do well to have on the table close at  
hand while he is reading."—SATURDAY REVIEW.

**MORRIS**—Works by the Rev. R. MORRIS, LL.D., Lecturer  
on English Language and Literature in King's College  
School.

*HISTORICAL OUTLINES OF ENGLISH ACCIDENCE*,  
comprising Chapters on the History and Development of the  
Language, and on Word-formation. Fourth Edition. Extra  
fcap. 8vo. 6s.

"It marks an era in the study of the English tongue."—SATURDAY  
REVIEW.

"A genuine and sound book."—ATHENÆUM.

*ELEMENTARY LESSONS IN HISTORICAL  
ENGLISH GRAMMAR*, containing Accidence and Word-  
formation. Third Edition. 18mo. 2s. 6d.

*PRIMER OF ENGLISH GRAMMAR*, 18mo. 1s.

**OLIPHANT—THE OLD AND MIDDLE ENGLISH.** A New Edition of *THE SOURCES OF STANDARD ENGLISH*. By J. KINGTON OLIPHANT. Extra fcap. 8vo. 6s.

**PALGRAVE—THE CHILDREN'S TREASURY OF LYRICAL POETRY.** Selected and Arranged with Notes by FRANCIS TURNER PALGRAVE. 18mo. 2s. 6d. Also in Two parts. 18mo. 1s. each.

"While indeed a treasure for intelligent children, it is also a work which many older folk will be glad to have."—SATURDAY REVIEW.

**PYLODET—NEW GUIDE TO GERMAN CONVERSATION:** containing an Alphabetical List of nearly 800 Familiar Words followed by Exercises, Vocabulary of Words in frequent use; Familiar Phrases and Dialogues; a Sketch of German Literature, Idiomatic Expressions, &c. By L. PYLODET. 18mo. cloth limp. 2s. 6d.

*A SYNOPSIS OF GERMAN GRAMMAR.* From the above. 18mo. 6d.

**READING BOOKS—**Adapted to the English and Scotch Codes. Bound in Cloth.

*PRIMER.* 18mo. (48 pp.) 2d.

*BOOK I.* for Standard I. 18mo. (96 pp.) 4d.

" II. " II. 18mo. (144 pp.) 5d.

" III. " III. 18mo. (160 pp.) 6d.

" IV. " IV. 18mo. (176 pp.) 8d.

" V. " V. 18mo. (380 pp.) 1s.

" VI. " VI. Crown 8vo. (430 pp.) 2s.

Book VI. is fitted for higher Classes, and as an Introduction to English Literature.

"They are far above any others that have appeared both in form and substance. . . . The editor of the present series has rightly seen that reading books must 'aim chiefly at giving to the pupils the power of accurate, and, if possible, apt and skilful expression; at cultivating in them a good literary taste, and at arousing a desire of further reading.' This is done by taking care to select the extracts from true English classics, going up in Standard VI. course to Chaucer, Hooker, and Bacon, as well as Wordsworth, Macaulay, and Froude. . . . This is quite on the right track, and indicates justly the ideal which we ought to set before us."—GUARDIAN.

**SHAKESPEARE—A SHAKESPEARE MANUAL.** By F. G. FLEAY, M.A., Head Master of Skipton Grammar School. Second Edition. Extra fcap. 8vo. 4s. 6d.

"A valuable contribution to the study of Shakespeare."—SATURDAY REVIEW.

**SHAKESPEARE** *continued*—

*AN ATTEMPT TO DETERMINE THE CHRONOLOGICAL ORDER OF SHAKESPEARE'S PLAYS.* By the Rev. H. PAINE STOKES, B.A. Extra fcap. 8vo. 4s. 6d.

**SKEAT**—*SHAKESPEARE'S PLUTARCH.* Being a Selection from the Lives in North's Plutarch which illustrate Shakespeare's Plays. Edited, with Introductions, Notes, Index of Names, and Glossarial Index, by the Rev. W. W. SKEAT, M.A. Crown 8vo. 6s.

**SONNENSCHN and MEIKLEJOHN**—*THE ENGLISH METHOD OF TEACHING TO READ.* By A. SONNENSCHN and J. M. D. MEIKLEJOHN, M.A. Fcap. 8vo.

## COMPRISING :

*THE NURSERY BOOK*, containing all the Two-Letter Words in the Language. 1d. (Also in Large Type on Sheets for School Walls. 5s.)

*THE FIRST COURSE*, consisting of Short Vowels with Single Consonants. 6d.

*THE SECOND COURSE*, with Combinations and Bridges. consisting of Short Vowels with Double Consonants. 6d.

*THE THIRD AND FOURTH COURSES*, consisting of Long Vowels, and all the Double Vowels in the Language. 6d.

"These are admirable books, because they are constructed on a principle, and that the simplest principle on which it is possible to learn to read English."—SPECTATOR.

**TANNER**—*FIRST PRINCIPLES OF AGRICULTURE.* By H. TANNER, F.C.S., Professor of Agricultural Science, University College, Aberystwith, &c. 18mo. 1s.

"We cordially recommend it to all students and teachers of this most important science."—SCHOOLMASTER.

**TAYLOR**—*WORDS AND PLACES*; or, Etymological Illustrations of History, Ethnology, and Geography. By the Rev ISAAC TAYLOR, M.A. Third and cheaper Edition, revised and compressed. With Maps. Globe 8vo. 6s.

**TAYLOR**—*A PRIMER OF PIANOFORTE PLAYING.* By FRANKLIN TAYLOR. Edited by GEORGE GROVE. 18mo. 1s.

"There are many hints of almost priceless worth not only to pupils but to teachers."—MORNING POST.

**TEGETMEIER**—*HOUSEHOLD MANAGEMENT AND COOKERY*. With an Appendix of Recipes used by the Teachers of the National School of Cookery. By W. B. TEGETMEIER. Compiled at the request of the School Board for London. 18mo. 1s.

"Admirably adapted to the use for which it was designed."—

**ATHENÆUM**

"A seasonable and thoroughly practical manual."—**PALL MALL GAZETTE**.

**THRING**—Works by EDWARD THRING, M.A., Head Master of Uppingham.

*THE ELEMENTS OF GRAMMAR TAUGHT IN ENGLISH*. With Questions. Fourth Edition. 18mo. 2s.

*THE CHILD'S GRAMMAR*. Being the Substance of "The Elements of Grammar taught in English," adapted for the Use of Junior Classes. A New Edition. 18mo. 1s.

*SCHOOL SONGS*. A Collection of Songs for Schools. With the Music arranged for four Voices. Edited by the Rev. E. THRING and H. RICCIUS. Folio. 7s. 6d.

**TRENCH (ARCHBISHOP)**—Works by R. C. TRENCH, D.D., Archbishop of Dublin.

*HOUSEHOLD BOOK OF ENGLISH POETRY*. Selected and Arranged, with Notes. Second Edition. Extra fcap. 8vo. 5s. 6d.

*ON THE STUDY OF WORDS*. Lectures addressed (originally) to the Pupils at the Diocesan Training School, Winchester. Seventeenth Edition, revised. Fcap. 8vo. 5s.

*ENGLISH, PAST AND PRESENT*. Tenth Edition, revised and improved. Fcap. 8vo. 5s.

*A SELECT GLOSSARY OF ENGLISH WORDS*, used formerly in Senses Different from their Present. Fourth Edition, enlarged. Fcap. 8vo. 4s. 6d.

**VAUGHAN (C. M.)**—*WORDS FROM THE POETS*. By C. M. VAUGHAN. Eighth Edition. 18mo. cloth. 1s.

**WHITNEY**—Works by WILLIAM D. WHITNEY, Professor of Sanskrit and Instructor in Modern Languages in Yale College; first President of the American Philological Association, and hon. member of the Royal Asiatic Society of Great Britain and Ireland; and Correspondent of the Berlin Academy of Sciences.

**WHITNEY Continued—**

*A COMPENDIOUS GERMAN GRAMMAR.* Crown 8vo. 4s. 6d.

*A GERMAN READER IN PROSE AND VERSE,* with Notes and Vocabulary. Crown 8vo. 5s.

**WHITNEY AND EDGREN—***A COMPENDIOUS GERMAN AND ENGLISH DICTIONARY,* with Notation of Correspondences and Brief Etymologies. By Professor W. D. WHITNEY, assisted by A. H. EDGREN. Crown 8vo. 7s. 6d.  
*THE GERMAN-ENGLISH,* separately, 5s.

**YONGE (CHARLOTTE M.)—***THE ABRIDGED BOOK OF GOLDEN DEEDS.* A Reading Book for Schools and general readers. By the Author of "The Heir of Redclyffe." 18mo. cloth. 1s.

## HISTORY.

**FREEMAN (EDWARD A.)—OLD-ENGLISH HISTORY.**

By EDWARD A. FREEMAN, D.C.L., LL.D., late Fellow of Trinity College, Oxford. With Five Coloured Maps. New Edition. Extra fcap. 8vo. half-bound. 6s.

"The book indeed is full of instruction and interest to students of all ages, and he must be a well-informed man indeed who will not rise from its perusal with clearer and more accurate ideas of a too much neglected portion of English History."—SPECTATOR.

**GREEN.—***A SHORT HISTORY OF THE ENGLISH PEOPLE.* By JOHN RICHARD GREEN. With Coloured Maps, Genealogical Tables, and Chronological Annals. Crown 8vo. 8s. 6d. Fifty-fifth Thousand.

"Stands alone as the one general history of the country, for the sake of which all others, if young and old are wise, will be speedily and surely set aside."—ACADEMY.

**HISTORICAL COURSE FOR SCHOOLS—**Edited by EDWARD A. FREEMAN, D.C.L., late Fellow of Trinity College, Oxford.

**I. GENERAL SKETCH OF EUROPEAN HISTORY.**  
By EDWARD A. FREEMAN, D.C.L. New Edition, revised and enlarged, with Chronological Table, Maps, and Index. 18mo. cloth. 3s. 6d.

**HISTORICAL COURSE FOR SCHOOLS, Continued—**

"It supplies the great want of a good foundation for historical teaching. The scheme is an excellent one, and this instalment has been executed in a way that promises much for the volumes that are yet to appear."—**EDUCATIONAL TIMES.**

**II. HISTORY OF ENGLAND.** By EDITH THOMPSON.  
New Edition, revised and enlarged, with Maps. 18mo. 2s. 6d.

"Freedom from prejudice, simplicity of style, and accuracy of statement are the characteristics of this little volume. It is a trustworthy text-book and likely to be generally serviceable in schools."—**PALL MALL GAZETTE.**

"Upon the whole, this manual is the best sketch of English history for the use of young people we have yet met with."—**ATHENÆUM.**

**III. HISTORY OF SCOTLAND.** By MARGARET MACARTHUR. New Edition. 18mo. 2s.

"An excellent summary, unimpeachable as to facts, and putting them in the clearest and most impartial light attainable."—**GUARDIAN.**

"Miss Macarthur has performed her task with admirable care, clearness, and fulness, and we have now for the first time a really good School History of Scotland."—**EDUCATIONAL TIMES.**

**IV. HISTORY OF ITALY.** By the Rev. W. HUNT, M.A.  
18mo. 3s.

"It possesses the same solid merit as its predecessors . . . the same scrupulous care about fidelity in details. . . . It is distinguished, too, by information on art, architecture, and social politics, in which the writer's grasp is seen by the firmness and clearness of his touch."—**EDUCATIONAL TIMES.**

**V. HISTORY OF GERMANY.** By J. SIME, M.A.  
18mo. 3s.

"A remarkably clear and impressive history of Germany. Its great events are wisely kept as central figures, and the smaller events are carefully kept, not only subordinate and subservient, but most skilfully woven into the texture of the historical tapestry presented to the eye."—**STANDARD.**

**VI. HISTORY OF AMERICA.** By JOHN A. DOYLE.  
With Maps. 18mo. 4s. 6d.

"Mr. Doyle has performed his task with admirable care, fulness, and clearness, and for the first time we have for schools an accurate and interesting history of America, from the earliest to the present time."—**STANDARD.**

**EUROPEAN COLONIES.** By E. J. PAYNE, M.A. With  
Maps. 18mo. 4s. 6d.

"We have seldom met with an historian capable of forming a more comprehensive, far-seeing, and unprejudiced estimate of events and peoples, and we can commend this little work as one certain to prove of the highest interest to all thoughtful readers."—**TIMES.**

The following is in preparation :—

**FRANCE.** By CHARLOTTE M. YONGE.

**HISTORY PRIMERS**—Edited by JOHN RICHARD GREEN.  
Author of "A Short History of the English People."

**ROME.** By the Rev. M. CREIGHTON, M.A., Fellow and Tutor of Merton College, Oxford. With Eleven Maps. New Edition. 18mo. 1s.

"The author has been curiously successful in telling in an intelligent way the story of Rome from first to last."—SCHOOL BOARD CHRONICLE.

**GREECE.** By C. A. FYFFE, M.A., Fellow and late Tutor of University College, Oxford. With Five Maps. New Edition. 18mo. 1s.

"We give our unqualified praise to this little manual."—SCHOOLMASTER.

**EUROPEAN HISTORY.** By E. A. FREEMAN, D.C.L., LL.D. With Maps. New Edition. 18mo. 1s.

"A marvel of clearness."—ACADEMY.

"The work is always clear, and forms a luminous key to European history."—SCHOOL BOARD CHRONICLE.

"There are few writers but himself who could have compressed so much information in so little space."—EDUCATIONAL TIMES.

**GREEK ANTIQUITIES.** By the Rev. J. P. MAHAFFY, M.A. Illustrated. 18mo. 1s.

"All that is necessary for the scholar to know is told so compactly yet so fully, and in a style so interesting, that it is impossible for even the dullest boy to look on this little work in the same light as he regards his other school books."—SCHOOLMASTER.

**CLASSICAL GEOGRAPHY.** By H. F. TOZER, M.A. 18mo. 1s.

"Another valuable aid to the study of the ancient world. . . . It contains an enormous quantity of information packed into a small space, and at the same time communicated in a very readable shape."—JOHN BULL.

**GEOGRAPHY.** By GEORGE GROVE, D.C.L. With Maps. 18mo. 1s.

"A model of what such a work should be . . . we know of no short treatise better suited to infuse life and spirit into the dull lists of proper names of which our ordinary class-books so often almost exclusively consist."—TIMES.

**ROMAN ANTIQUITIES.** By Professor WILKINS. Illustrated. 18mo. 1s.

"A little book that throws a blaze of light on Roman History, and is, moreover, intensely interesting."—School Board Chronicle.



**HISTORY PRIMERS** *Continued—*

In preparation :—

*ENGLAND.* By J. R. GREEN, M.A.

*FRANCE.* By CHARLOTTE M. YONGE.

**MICHELET—A SUMMARY OF MODERN HISTORY.**

Translated from the French of M. MICHELET, and continued to the Present Time, by M. C. M. SIMPSON. Globe 8vo. 4s. 6d.

"We are glad to see one of the ablest and most useful summaries of European history put into the hands of English readers. The translation is excellent."—STANDARD.

**OTTÉ—SCANDINAVIAN HISTORY.** By E. C. OTTÉ.

With Maps. Globe 8vo. 6s.

"A readable, well-arranged, complete, and accurate volume."—LITERARY REVIEW.

**PAULI.—PICTURES OF OLD ENGLAND.** By Dr. R.

PAULI. Translated with the sanction of the Author by E. C. OTTÉ. Cheaper Edition. Crown 8vo. 6s.

**YONGE (CHARLOTTE M.)—A PARALLEL HISTORY OF**

*FRANCE AND ENGLAND*: consisting of Outlines and Dates. By CHARLOTTE M. YONGE, Author of "The Heir of Redclyffe," "Cameos of English History," &c., &c. Oblong 4to. 3s. 6d.

"We can imagine few more really advantageous courses of historical study for a young mind than going carefully and steadily through Miss Yonge's excellent little book."—EDUCATIONAL TIMES.

*CAMEOS FROM ENGLISH HISTORY. — FROM ROLLO TO EDWARD II.* By the Author of "The Heir of Redclyffe." Extra fcap. 8vo. Third Edition, enlarged. 5s.

"Instead of dry details, we have living pictures, faithful, vivid, and striking."—NONCONFORMIST.

*A SECOND SERIES OF CAMEOS FROM ENGLISH HISTORY—THE WARS IN FRANCE.* Third Edition.

Extra fcap. 8vo. 5s.

"Though mainly intended for young readers, they will, if we mistake not, be found very acceptable to those of more mature years, and the life and reality imparted to the dry bones of history cannot fail to be attractive to readers of every age."—JOHN BULL.

**YONGE, CHARLOTTE M.,** *Continued—*

*A THIRD SERIES OF CAMEOS FROM ENGLISH HISTORY—THE WARS OF THE ROSES.* Extra fcap. 8vo. 5s.

*A FOURTH SERIES.*

[*In the press.*]

*EUROPEAN HISTORY.* Narrated in a Series of Historical Selections from the Best Authorities. Edited and arranged by E. M. SEWELL and C. M. YONGE. First Series, 1003—1154. Third Edition. Crown 8vo. 6s. Second Series, 1088—1228. Third Edition. Crown 8vo. 6s.

"We know of scarcely anything which is so likely to raise to a higher level the average standard of English education."—*GUARDIAN.*

## DIVINITY.

\*.\* For other Works by these Authors, see *THEOLOGICAL CATALOGUE.*

**ABBOTT (REV. E. A.)—BIBLE LESSONS.** By the Rev. E. A. ABBOTT, D.D., Head Master of the City of London School. Second Edition. Crown 8vo. 4s. 6d.

"Wise, suggestive, and really profound initiation into religious thought."  
—*GUARDIAN.*

"I think nobody could read them without being both the better for them himself, and being also able to see how this difficult duty of imparting a sound religious education may be effected."—*BISHOP OF ST. DAVID'S AT ABERGWILLY.*

**ARNOLD—A BIBLE-READING FOR SCHOOLS—THE GREAT PROPHECY OF ISRAEL'S RESTORATION** (Isaiah, Chapters xl.—lxvi.). Arranged and Edited for Young Learners. By MATTHEW ARNOLD, D.C.L., formerly Professor of Poetry in the University of Oxford, and Fellow of Oriel. Fourth Edition. 18mo. cloth. 1s.

"There can be no doubt that it will be found excellently calculated to further instruction in Biblical literature in any school into which it may be introduced; and we can safely say that whatever school uses the book, it will enable its pupils to understand Isaiah, a great advantage compared with other establishments which do not avail themselves of it."—*TIMES.*

*ISAIAH XL.—LXVI.* With the Shorter Prophecies allied to it. Arranged and Edited, with Notes, by MATTHEW ARNOLD. Crown 8vo. 5s.

**GOLDEN TREASURY PSALTER**—Students' Edition. Being an Edition of "The Psalms Chronologically Arranged, by Four Friends," with briefer Notes. 18mo. 3s. 6d.

**HARDWICK**—Works by Archdeacon HARDWICK.

*A HISTORY OF THE CHRISTIAN CHURCH.* Middle Age. From Gregory the Great to the Excommunication of Luther. Edited by WILLIAM STUBBS, M.A., Regius Professor of Modern History in the University of Oxford. With Four Maps constructed for this work by A. KEITH JOHNSTON. Fourth Edition. Crown 8vo. 10s. 6d.

"As a manual for the student of ecclesiastical history in the Middle Ages, we know no English work which can be compared to Mr. Hardwick's book."—GUARDIAN.

*A HISTORY OF THE CHRISTIAN CHURCH DURING THE REFORMATION.* Fourth Edition. Edited by Professor STUBBS. Crown 8vo. 10s. 6d.

**MACLEAR**—Works by the Rev. G. F. MACLEAR, D.D., Head Master of King's College School.

*A CLASS-BOOK OF OLD TESTAMENT HISTORY.* New Edition, with Four Maps. 18mo. 4s. 6d.

"A careful and elaborate though brief compendium of all that modern research has done for the illustration of the Old Testament. We know of no work which contains so much important information in so small a compass."—BRITISH QUARTERLY REVIEW.

*A CLASS-BOOK OF NEW TESTAMENT HISTORY,* including the Connection of the Old and New Testament. With Four Maps. New Edition. 18mo. 5s. 6d.

"A singularly clear and orderly arrangement of the Sacred Story. His work is solidly and completely done."—ATHENÆUM.

*A SHILLING BOOK OF OLD TESTAMENT HISTORY,* for National and Elementary Schools. With Map. 18mo. cloth. New Edition.

*A SHILLING BOOK OF NEW TESTAMENT HISTORY,* for National and Elementary Schools. With Map. 18mo. cloth. New Edition.

These works have been carefully abridged from the author's larger manuals.

**MACLEAR** *Continued*—

*CLASS-BOOK OF THE CATECHISM OF THE CHURCH OF ENGLAND.* New Edition. 18mo. cloth. 1s. 6d.

"It is indeed the work of a scholar and divine, and as such, though extremely simple, it is also extremely instructive. There are few clergymen who would not find it useful in preparing candidates for Confirmation; and there are not a few who would find it useful to themselves as well."—  
LITERARY CHURCHMAN.

*A FIRST CLASS-BOOK OF THE CATECHISM OF THE CHURCH OF ENGLAND,* with Scripture Proofs, for Junior Classes and Schools. 18mo. 6d. New Edition.

*A<sup>1</sup> MANUAL OF INSTRUCTION FOR CONFIRMATION AND FIRST COMMUNION. WITH PRAYERS AND DEVOTIONS.* 32mo. cloth extra, red edges. 2s.

"It is earnest, orthodox, and affectionate in tone. The form of self-examination is particularly good."—JOHN BULL.

*THE ORDER OF CONFIRMATION, WITH PRAYERS AND DEVOTIONS.* 32mo. 6d.

*FIRST COMMUNION, WITH PRAYERS AND DEVOTIONS FOR THE NEWLY CONFIRMED.* 32mo. 6d.

**M'CLELLAN**—*THE NEW TESTAMENT.* A New Translation on the Basis of the Authorised Version, from a Critically revised Greek Text, with Analyses, copious References and Illustrations from original authorities, New Chronological and Analytical Harmony of the Four Gospels, Notes and Dissertations. A contribution to Christian Evidence. By JOHN BROWN M'CLELLAN, M.A., late Fellow of Trinity College, Cambridge. In Two Vols. Vol. I.—The Four Gospels with the Chronological and Analytical Harmony. 8vo. 30s.

"One of the most remarkable productions of recent times," says the *Theological Review*, "in this department of sacred literature;" and the *British Quarterly Review* terms it "a thesaurus of first-hand investigations."

**MAURICE**—*THE LORD'S PRAYER, THE CREED, AND THE COMMANDMENTS.* A Manual for Parents and Schoolmasters. To which is added the Order of the Scriptures. By the Rev. F. DENISON MAURICE, M.A. 18mo. cloth, limp. 1s.

**PROCTER**—*A HISTORY OF THE BOOK OF COMMON PRAYER*, with a Rationale of its Offices. By FRANCIS PROCTER, M.A. Thirteenth Edition, revised and enlarged. Crown 8vo. 10s. 6d.

**PROCTER AND MACLEAR**—*AN ELEMENTARY INTRODUCTION TO THE BOOK OF COMMON PRAYER*. Re-arranged and supplemented by an Explanation of the Morning and Evening Prayer and the Litany. By the Rev. F. PROCTER and the Rev. Dr. MACLEAR. New and Enlarged Edition, containing the Communion Service and the Confirmation and Baptismal Offices. 18mo. 2s. 6d.

**PSALMS OF DAVID CHRONOLOGICALLY ARRANGED.**  
By Four Friends. An Amended Version, with Historical Introduction and Explanatory Notes. Second and Cheaper Edition, with Additions and Corrections. Crown 8vo. 8s. 6d.

"One of the most instructive and valuable books that has been published for many years."—SPECTATOR.

**RAMSAY**—*THE CATECHISER'S MANUAL*; or, the Church Catechism Illustrated and Explained, for the Use of Clergymen, Schoolmasters, and Teachers. By the Rev. ARTHUR RAMSAY, M.A. Second Edition. 18mo. 1s. 6d.

**SIMPSON**—*AN EPITOME OF THE HISTORY OF THE CHRISTIAN CHURCH*. By WILLIAM SIMPSON, M.A. Fifth Edition. Fcap. 8vo. 3s. 6d.

**TRENCH**—By R. C. TRENCH, D.D., Archbishop of Dublin. *LECTURES ON MEDIEVAL CHURCH HISTORY*. Being the substance of Lectures delivered at Queen's College, London. 8vo. 12s.

*SYNONYMS OF THE NEW TESTAMENT*. Eighth Edition, revised. 8vo. 12s.

**WESTCOTT**—Works by BROOKE FOSS WESTCOTT, D.D., Canon of Peterborough.

*A GENERAL SURVEY OF THE HISTORY OF THE CANON OF THE NEW TESTAMENT DURING THE FIRST FOUR CENTURIES.* Fourth Edition. With Preface on "Supernatural Religion." Crown 8vo. 10s. 6d.

"As a theological work it is at once perfectly fair and impartial, and imbued with a thoroughly religious spirit; and as a manual it exhibits, in a lucid form and in a narrow compass, the results of extensive research and accurate thought. We cordially recommend it."—SATURDAY REVIEW.

*INTRODUCTION TO THE STUDY OF THE FOUR GOSPELS.* Fifth Edition. Crown 8vo. 10s. 6d.

"To learning and accuracy which commands respect and confidence, he unites what are not always to be found in union with these qualities, the no less valuable faculties of lucid arrangement and graceful and facile expression."—LONDON QUARTERLY REVIEW.

*THE BIBLE IN THE CHURCH.* A Popular Account of the Collection and Reception of the Holy Scriptures in the Christian Churches. New Edition. 18mo. cloth. 4s. 6d.

"We would recommend every one who loves and studies the Bible to read and ponder this exquisite little book. Mr. Westcott's account of the 'Canon' is *true history* in its highest sense."—LITERARY CHURCHMAN.

*THE GOSPEL OF THE RESURRECTION.* Thoughts on its Relation to Reason and History. New Edition. Crown 8vo. 6s.

**WILSON**—*THE BIBLE STUDENT'S GUIDE* to the more Correct Understanding of the English Translation of the Old Testament, by reference to the original Hebrew. By WILLIAM WILSON, D.D., Canon of Winchester, late Fellow of Queen's College, Oxford. Second Edition, carefully revised. 4to. cloth. 25s.

"For all earnest students of the Old Testament Scriptures it is a most valuable manual. Its arrangement is so simple that those who possess only their mother-tongue, if they will take a little pains, may employ it with great profit."—NONCONFORMIST.

**YONGE (CHARLOTTE M.)—SCRIPTURE READINGS FOR SCHOOLS AND FAMILIES.** By CHARLOTTE M. YONGE, Author of "The Heir of Redclyffe."

FIRST SERIES. GENESIS TO DEUTERONOMY. Globe 8vo. 1s. 6d. With Comments, Second Edition, 3s. 6d.

SECOND SERIES. From JOSHUA to SOLOMON. Extra fcap. 8vo. 1s. 6d. With Comments, 3s. 6d.

THIRD SERIES. The KINGS and the PROPHETS. Extra fcap. 8vo. 1s. 6d. With Comments, 3s. 6d.

FOURTH SERIES. The GOSPEL TIMES. 1s. 6d. With Comments, extra fcap. 8vo., 3s. 6d. ;

FIFTH SERIES.

[*In the press.*]

Actual need has led the author to endeavour to prepare a reading book convenient for study with children, containing the very words of the Bible, with only a few expedient omissions, and arranged in Lessons of such length as by experience she has found to suit with children's ordinary power of accurate attentive interest. The verse form has been retained, because of its convenience for children reading in class, and as more resembling their Bibles; but the poetical portions have been given in their lines. When Psalms or portions from the Prophets illustrate or fall in with the narrative they are given in their chronological sequence. The Scripture portion, with a very few notes explanatory of mere words, is bound up apart, to be used by children, while the same is also supplied with a brief comment, the purpose of which is either to assist the teacher in explaining the lesson, or to be used by more advanced young people to whom it may not be possible to give access to the authorities whence it has been taken. Professor Huxley, at a meeting of the London School Board, particularly mentioned the selection made by Miss Yonge as an example of how selections might be made from the Bible for School Reading. See TIMES, March 30, 1871.

## MACMILLAN'S GLOBE LIBRARY.

Beautifully printed on toned paper, price 3s. 6d. each. Also kept in various morocco and calf bindings, at moderate prices.

The *Saturday Review* says :—"The Globe Editions are admirable for their scholarly editing, their typographical excellence, their compendious form, and their cheapness."

The *Daily Telegraph* calls it "a series yet unrivalled for its combination of excellence and cheapness."

**SHAKESPEARE'S COMPLETE WORKS.** Edited by W. G. CLARK, M.A., and W. ALDIS WRIGHT, M.A. With Glossary.

**MORTE D'ARTHUR.** Sir Thomas Malory's Book of King Arthur and of his Noble Knights of the Round Table. The Edition of Caxton, revised for Modern Use. With an Introduction, Notes, and Glossary, by Sir EDWARD STRACHEY.

**BURNS'S COMPLETE WORKS:** the Poems, Songs, and Letters. Edited, with Glossarial Index and Biographical Memoir, by ALEXANDER SMITH.

**ROBINSON CRUSOE.** Edited, after the Original Editions, with Biographical Introduction, by HENRY KINGSLEY.

**SCOTT'S POETICAL WORKS.** With Biographical and Critical Essay, by FRANCIS TURNER PALGRAVE.

**GOLDSMITH'S MISCELLANEOUS WORKS.** With Biographical Introduction by Professor MASSON.

**SPENSER'S COMPLETE WORKS.** Edited, with Glossary, by R. MORRIS, and Memoir by J. W. HALES.

**POPE'S POETICAL WORKS.** Edited, with Notes and Introductory Memoir, by Professor WARD.

**DRYDEN'S POETICAL WORKS.** Edited, with a Revised Text and Notes, by W. D. CHRISTIE, M.A., Trinity College, Cambridge.

**COWPER'S POETICAL WORKS.** Edited, with Notes and Biographical Introduction, by W. BENHAM.

**VIRGIL'S WORKS.** Rendered into English Prose. With Introductions, Notes, Analysis, and Index, by J. LONSDALE, M.A., and S. LEE, M.A.

**HORACE.** Rendered into English Prose. With running Analysis, Introduction, and Notes, by J. LONSDALE, M.A., and S. LEE, M.A.

**MILTON'S POETICAL WORKS.** Edited, with Introductions, &c., by Professor MASSON.



*Published every Thursday, price 6d.; Monthly Parts, 2s.  
and 2s. 6d., Half-Yearly Volumes, 15s.*

# NATURE:

*AN ILLUSTRATED JOURNAL OF SCIENCE.*

NATURE expounds in a popular and yet authentic manner, the GRAND RESULTS OF SCIENTIFIC RESEARCH, discussing the most recent scientific discoveries, and pointing out the bearing of Science upon civilisation and progress, and its claims to a more general recognition, as well to a higher place in the educational system of the country.

It contains original articles on all subjects within the domain of Science; Reviews setting forth the nature and value of recent Scientific Works; Correspondence Columns, forming a medium of Scientific discussion and of intercommunication among the most distinguished men of Science; Serial Columns, giving the gist of the most important papers appearing in Scientific Journals, both Home and Foreign; Transactions of the principal Scientific Societies and Academies of the World; Notes, &c.

In schools where Science is included in the regular course of studies, this paper will be most acceptable, as it tells what is doing in Science all over the world, is popular without lowering the standard of Science, and by it a vast amount of information is brought within a small compass, and students are directed to the best sources for what they need. The various questions connected with Science teaching in schools are also fully discussed, and the best methods of teaching are indicated.



